

QUOTIENTS, AUTOMORPHISMS AND DIFFERENTIAL OPERATORS

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ABSTRACT. Let V be a G -module where G is a complex reductive group. Let $Z := V//G$ denote the categorical quotient and let $\pi: V \rightarrow Z$ be the morphism dual to the inclusion $\mathcal{O}(V)^G \subset \mathcal{O}(V)$. Let $\varphi: Z \rightarrow Z$ be an algebraic automorphism. Then one can ask if there is an algebraic map $\Phi: V \rightarrow V$ which lifts φ , i.e., $\pi(\Phi(v)) = \varphi(\pi(v))$ for all $v \in V$. In Kuttler [Kut11] the case is treated where $V = r\mathfrak{g}$ is a multiple of the adjoint representation of G . It is shown that, for r sufficiently large (often $r \geq 2$ will do), any φ has a lift.

We consider the case of general representations. It turns out that it is natural to consider holomorphic lifting of holomorphic automorphisms of Z , and we show that if a holomorphic φ lifts holomorphically, then it has a lift Φ which is an automorphism such that $\Phi(gv) = \sigma(g)\Phi(v)$, $v \in V$, $g \in G$ where σ is an automorphism of G . Lifting does not always hold, but we show that it always does for representations of tori in which case algebraic automorphisms lift to algebraic automorphisms. We extend Kuttler's methods to show lifting in case V contains a copy of \mathfrak{g} .

1. INTRODUCTION

Our base field is \mathbb{C} , the field of complex numbers. Let G be a complex reductive group and V a G -module. We denote the algebra of polynomial functions on V by $\mathcal{O}(V)$. For the following, we refer to [Kra84], [VP89] and [Lun73]. By Hilbert, the algebra $\mathcal{O}(V)^G$ is finitely generated, so that we have a quotient variety $Z := V//G$ with coordinate ring $\mathcal{O}(Z) = \mathcal{O}(V)^G$. Let $\pi: V \rightarrow Z$ denote the morphism dual to the inclusion $\mathcal{O}(V)^G \subset \mathcal{O}(V)$. Then π sets up a bijection between the points of Z and the closed orbits in V . If Gv is a closed orbit, then the isotropy group $H = G_v$ is reductive. The *slice representation of H at v* is its action on N_v where N_v is an H -complement to $T_v(Gv)$ in $T_v(V) \simeq V$. Let $Z_{(H)}$ denote the points of Z such that the isotropy groups of the corresponding closed orbits are in the conjugacy class (H) of H . The $Z_{(H)}$ give a finite stratification of Z by locally closed smooth subvarieties. In particular, there is a unique open stratum $Z_{(H)}$, the *principal stratum*, which we also denote by Z_{pr} . We call H a *principal isotropy group* and any associated closed orbit a *principal orbit* of G .

As shorthand for saying that V has finite (resp. trivial) principal isotropy groups we say that V has FPIG (resp. TPIG). If V has FPIG, then there is an open set of closed orbits and a closed orbit is principal if and only if the slice representation of its isotropy group is trivial. Set $V_{\text{pr}} := \pi^{-1}(Z_{\text{pr}})$. We say that V is *k-principal* if V has FPIG and $\text{codim } V \setminus V_{\text{pr}} \geq k$.

Let $\mathcal{D}^k(V)$ (resp. $\mathcal{D}^k(Z)$) denote the differential operators on V (resp. Z) of order at most k (see [Sch95, §3]). Then restriction gives us a morphism $\pi_*: \mathcal{D}^k(V)^G \rightarrow \mathcal{D}^k(Z)$. One just considers elements of $\mathcal{D}^k(V)^G$ as differential operators on $\mathcal{O}(V)^G = \mathcal{O}(Z)$.

Definition 1.1. We say that V is *admissible* if

- (1) V is 2-principal.
- (2) $\pi_*: \mathcal{D}^k(V) \rightarrow \mathcal{D}^k(Z)$ is surjective for all k .

If V is 2-principal, then the principal isotropy group is the kernel of $G \rightarrow \text{GL}(V)$ [Sch13, Remark 2.5]. Thus if V is admissible, we can assume that it has TPIG by just dividing out by the ineffective part of the action. If G^0 is a torus, then (1) implies (2) (see [Sch95, 10.4]).

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Let \mathfrak{g} denote the Lie algebra of G . Suppose that $G = G_1 \times G_2$ is a product of reductive groups. Consider representations $V = V_1 \oplus V_2$ where V_i is a representation of G_i , $i = 1, 2$. If V_1 is not admissible, then neither is $V_1 \oplus V_2$. This explains why the hypotheses of the following theorem are necessary.

Theorem 1.2. [Sch95, Corollary 11.6] *Let G be a connected semisimple group and consider representations of G which contain no trivial factor and all of whose irreducible factors are faithful representations of \mathfrak{g} . Then, up to isomorphism, there are only finitely many such representations which are not admissible.*

For X an affine variety let $\text{Aut}(X)$ denote the automorphisms of X and let $\text{Aut}_{\mathcal{H}}(X)$ denote the holomorphic automorphisms of X . Let $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$. We say that a holomorphic map $\Phi: V \rightarrow V$ is a *lift* of φ if $\pi \circ \Phi = \varphi \circ \pi$. Equivalently, Φ maps the fiber $\pi^{-1}(z)$ over $z \in Z$ to the fiber $\pi^{-1}(\varphi(z))$. Let σ be an automorphism of G . We say that Φ is σ -equivariant if $\Phi(gv) = \sigma(g)\Phi(v)$ for all $v \in V$, $g \in G$. Let $\text{Aut}_{\text{qel}}(Z)$ denote the set of *quasilinear automorphisms* of Z , i.e., the automorphisms of Z which preserve the grading of $\mathcal{O}(Z) \simeq \mathcal{O}(V)^G$. This is a linear algebraic group. If Φ is a holomorphic lift of $\varphi \in \text{Aut}_{\text{qel}}(Z)$, then so is $\Phi'(0)$, hence φ has a linear lift. When V is 2-principal we will see that $\Phi'(0)$ is invertible and normalizes G , hence induces an automorphism σ .

Theorem 1.3. *Let V be an admissible G -module and $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$. Then there is a holomorphic family φ_t with φ_0 quasilinear and $\varphi_1 = \varphi$. There are biholomorphic equivariant lifts Ψ_t of $\varphi_0^{-1}\varphi_t$, $t \in [0, 1]$. If φ_0 has a lift to V , then φ_0 has a σ -equivariant linear lift $\Phi_0 \in \text{GL}(V)$ for some σ and $\Phi_0\Psi_1$ is a σ -equivariant biholomorphic lift of φ . Finally, any element of $\text{Aut}_{\text{qel}}(Z)^0$ has a lift to $\text{GL}(V)$.*

Remark 1.4. In general, we do not know if we can lift algebraic automorphisms of Z algebraically. Our intuition is that this does not occur. By Theorem 1.10 below we can lift algebraic automorphisms algebraically in the case that G^0 is a torus.

Remark 1.5. In case G is finite, the lifting problem has been considered in [Got69, KLM03, LMP03, Pri67].

Remark 1.6. Suppose that $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$ has a lift to V . Then there is a holomorphic family φ_t as above such that φ_0 lifts to V (Proposition 2.1 below). Theorem 1.3 shows that φ has a biholomorphic σ -equivariant lift. Thus φ sends the stratum with conjugacy class (H) to the stratum with conjugacy class $(\sigma(H))$, i.e., φ permutes the strata of Z according to the action of σ on conjugacy classes. Kuttler [Kut11] found this to be true for the case of multiples of the adjoint representation by different methods.

Remark 1.7. If V is admissible, then [Sch13, Remark 2.4] shows that any $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$ permutes the strata of Z . This also remains true if V is orthogonal and 2-principal [Sch13, Theorems 1.1, 1.3]. When φ has a lift we get the stronger assertion above on how φ permutes the strata of Z .

From the theorem above it is clear that the lifting problem reduces to one for the connected components of $\text{Aut}_{\text{qel}}(Z)$. We say that V has the *lifting property* if every element of $\text{Aut}_{\text{qel}}(Z)$ lifts to $\text{GL}(V)$. In the cases of the classical representations of the classical groups (and a bit more) we can say exactly what happens. In the theorem below the actions of \mathbf{G}_2 (resp. $\mathbf{B}_3 = \text{Spin}_7$) on \mathbb{C}^7 (resp. \mathbb{C}^8) are the irreducible representations of the given dimension.

Theorem 1.8. *Let (V, G) be one of the following representations. Then V is admissible and V has the lifting property.*

- (1) $(k\mathbb{C}^n \oplus \ell(\mathbb{C}^n)^*, \text{SL}_n)$, $n \geq 3$ where $k + \ell \geq 2n$ and if $k\ell = 0$, then $k + \ell > 2n$.
- (2) $(k\mathbb{C}^n \oplus \ell(\mathbb{C}^n)^*, \text{GL}_n)$ where $k, \ell > n$.

- (3) $(k\mathbb{C}^n, \mathrm{SO}_n)$ where $k \geq n \geq 3$.
- (4) $(k\mathbb{C}^n, \mathrm{O}_n)$ where $k > n \geq 3$.
- (5) $(k\mathbb{C}^{2n}, \mathrm{Sp}_{2n})$ where $k \geq 2n + 2$, $n \geq 2$.
- (6) $(k\mathbb{C}^7, \mathrm{G}_2)$ where $k > 4$.
- (7) $(k\mathbb{C}^8, \mathrm{B}_3)$ where $k > 5$.

Remark 1.9. The conditions in the theorem are mostly those needed to guarantee that the representations are admissible (see [Sch95, §11]). However, (V, G) is admissible and we have an element of $\mathrm{Aut}_{\mathrm{qel}}(Z)$ which does not lift in the following cases

- (1) $(2n\mathbb{C}^n, \mathrm{SL}_n)$ and $(2n(\mathbb{C}^n)^*, \mathrm{SL}_n)$, $n \geq 2$.
- (2) $(4\mathbb{C}^7, \mathrm{G}_2)$.
- (3) $(5\mathbb{C}^8, \mathrm{B}_3)$.

If G^0 is a torus there is no problem in lifting.

Theorem 1.10. *Suppose that V is 2-principal and that G^0 is a torus. Then every algebraic (resp. holomorphic) automorphism of Z lifts to an algebraic (resp. holomorphic) automorphism of V which is σ -equivariant for some σ .*

We now consider a strengthening of this theorem. Let G be reductive and write $G^0 = G_s S$ where G_s is semisimple and S is the connected center of G^0 . Assume that V is 4-principal and let $\varphi \in \mathrm{Aut}(Z)$. Then (see Proposition 5.3) φ lifts to an automorphism of $Z_s := V//G_s$, so the lifting problem rests entirely in the action of G_s .

Using ideas of Kuttler [Kut11] we establish the following result.

Theorem 1.11. *Suppose that V is 4-principal and (V, G_s) contains a copy of \mathfrak{g}_s . Then any $\varphi \in \mathrm{Aut}(Z)$ has a lift to V .*

The following result is quite surprising. It says that Z often determines V and G .

Corollary 1.12. *Let $G_i \subset \mathrm{GL}(V_i)$ where each (V_i, G_i) satisfies the conditions of Theorem 1.11. Let Z_i denote $V_i//G_i$ and suppose that there is an algebraic isomorphism $\psi: Z_1 \rightarrow Z_2$. Then there is a linear isomorphism $V_1 \simeq V_2$ inducing an isomorphism of G_1 with G_2 .*

Proof. Let $V = V_1 \oplus V_2$ and $G = G_1 \times G_2$. Then $V//G \simeq Z_1 \times Z_2$ and (V, G) satisfies the hypotheses of Theorem 1.11. We have an automorphism φ of $Z_1 \times Z_2$ which sends (z_1, z_2) to $(\psi^{-1}(z_2), \psi(z_1))$. By Theorem 1.11 and Remark 1.6 this automorphism lifts to a holomorphic automorphism $\Phi: V \rightarrow V$ such that $\Phi(gv) = \sigma(g)\Phi(v)$ for $g \in G$, $v \in V$ where σ is an automorphism of $G_1 \times G_2$. Since Φ sends closed orbits to closed orbits and lies over φ , it sends $(V_1)_{\mathrm{pr}} \times (0)$ isomorphically onto $(0) \times (V_2)_{\mathrm{pr}}$, hence it sends $V_1 \times (0)$ isomorphically onto $(0) \times V_2$ and clearly σ has to interchange G_1 and G_2 . Now $\Phi'(0)$ induces a σ -equivariant linear isomorphism of V_1 with V_2 . \square

Remark 1.13. Of course, a similar corollary holds for 2-principal actions where G^0 a torus. In this case we get the desired result even if there is only a holomorphic isomorphism of Z_1 and Z_2 .

We generalize the results of Kuttler for multiples of the adjoint representation of a semisimple group and obtain a best possible result. Let $V = \bigoplus_{i=1}^d r_i \mathfrak{g}_i$ where the \mathfrak{g}_i are simple Lie algebras. Let G_i denote the adjoint group of \mathfrak{g}_i and let G denote the product of the G_i . We assume that $r_i \geq 2$ for all i and that $r_i \geq 3$ if $\mathfrak{g}_i \simeq \mathfrak{sl}_2$. These conditions are necessary and sufficient for (V, G) to be 2-principal and for any $\varphi \in \mathrm{Aut}(Z)$ to be strata preserving [Sch13, Theorem 1.2, Proposition 3.1]. The argument of Theorem 1.11 does not apply if any $r_i \mathfrak{g}_i$ is $3\mathfrak{sl}_2$, $4\mathfrak{sl}_2$, $2\mathfrak{sl}_3$, $3\mathfrak{sl}_3$, $2\mathfrak{o}_5$ or $2\mathfrak{sl}_4$. Studying these exceptional cases we are able to prove the following.

Theorem 1.14. *Let $V = \bigoplus_{i=1}^d r_i \mathfrak{g}_i$ be as above. Then any $\varphi \in \text{Aut}(Z)$ lifts to V .*

Note that we don't claim lifting for holomorphic automorphisms of Z . That is because we have not established that V is admissible, but we are sure that it is. We have checked this for simple groups of rank at most two.

Our results lead to consequences for actions of compact Lie groups (see [Sch80]). Let K be a compact Lie group and W a real K -module. Let W/K be the orbit space where we say that a function $f: W/K \rightarrow \mathbb{R}$ is C^∞ if it pulls back to an element of $C^\infty(W)$ (necessarily K -invariant). Then we can define the notion of a smooth automorphism of W/K . (See §8 for more details.) Let $G = K_{\mathbb{C}}$ be the complexification of K and set $V = W \otimes_{\mathbb{R}} \mathbb{C}$. We say that W is *admissible* if V is 2-principal.

Theorem 1.15. *Let W be an admissible K -module and let $\varphi: W/K \rightarrow W/K$ be a smooth automorphism. If V has the lifting property, then there is a lift $\Phi \in \text{Diff}(W)$ of φ and a group automorphism σ of K such that $\Phi(kw) = \sigma(k)\Phi(w)$ for every $k \in K$ and $w \in W$.*

It is possible that W has the lifting property (obvious definition) even if V does not, but we don't know of an example. Results on differentiable lifting in the case of finite groups can be found in [Bie75, Los01, Str82].

Here is an outline of the paper. In §2 we consider when an automorphism φ of Z can be deformed to a quasilinear automorphism. In §3 we establish Theorem 1.3 on holomorphic equivariant liftings. In §4 we establish Theorem 1.8 and Remark 1.9 (our examples). In §5 we establish Theorem 1.10, the result on actions of tori, in §6 we establish Theorem 1.11 on lifting for representations containing \mathfrak{g} . In §7 we consider the generalized adjoint case and establish Theorem 1.14 and in §8 we establish Theorem 1.15 on the actions of compact groups.

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2. DEFORMABLE AND QUASILINEAR AUTOMORPHISMS

We begin with some elementary remarks. Let V be a G -module. Then $\mathcal{O}(V)^G$ inherits a grading from $\mathcal{O}(V)$, hence we have a \mathbb{C}^* -action on $V//G$ where $t \cdot \pi(v) = \pi(tv)$. One can see this more concretely as follows. Let p_1, \dots, p_d be homogeneous generators of $\mathcal{O}(V)^G$ and let $p = (p_1, \dots, p_d): V \rightarrow \mathbb{C}^d$. Let Y denote the image of p . Then we can identify Y with $Z = V//G$. If e_i is the degree of p_i , then we have a \mathbb{C}^* -action on Y where $t \in \mathbb{C}^*$ sends $(y_1, \dots, y_d) \in Y$ to $(t^{e_1}y_1, \dots, t^{e_d}y_d)$. Suppose that $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$. We say that φ is *deformable* if $\varphi_0(z) := \lim_{t \rightarrow 0} t^{-1} \cdot \varphi(t \cdot z)$ exists for every $z \in Z$. Note that if all the p_i have the same degree, then the limit exists and is just an ordinary derivative. Also, if φ_0 exists, it is quasilinear.

Proposition 2.1. *Suppose that $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$ has a holomorphic lift $\Phi: V \rightarrow V$. Then φ is deformable to $\varphi_0 \in \text{Aut}_{\text{ql}}(Z)$ and φ_0 has a linear lift to V .*

Proof. Set $\Phi_t(v) = t^{-1}\Phi(tv)$ for $t \in \mathbb{C}^*$, $v \in V$. Then Φ_t covers φ_t where $\varphi_t(z) = t^{-1} \cdot \varphi(t \cdot z)$, $z \in Z$. Let $z = \pi(v) \in Z$. Then

$$\varphi_0(z) := \lim_{t \rightarrow 0} t^{-1} \cdot \varphi(t \cdot z) = \pi(\lim_{t \rightarrow 0} t^{-1}\Phi(tv)) = \pi(\Phi'(0)(v))$$

Hence φ is deformable to φ_0 and φ_0 lifts to $\Phi'(0)$. \square

Let $S = \bigoplus S_k$ denote the graded ring $\mathcal{O}(Z)$ and let $F_k = S_k + S_{k+1} + \dots$. We say that a differential operator $P \in \mathcal{D}(Z) = \bigoplus \mathcal{D}^k(Z)$ is *homogeneous of degree ℓ* if it sends elements of S_k to $S_{k+\ell}$ for all k . Let $\mathcal{H}(Z)$ (resp. $\mathcal{H}(V)$) denote the holomorphic functions on Z (resp. V).

- Remarks 2.2.* (1) Let $\varphi \in \text{Aut}(Z)$. Then φ is deformable if and only if $\varphi^* S_k \subset F_k$ for all $k \geq 0$. In this case the family φ_t , $t \in \mathbb{C}^*$, extends to an algebraic family φ_t , $t \in \mathbb{C}$.
- (2) Let $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$. Then φ is deformable if and only if $\varphi^* S_k \subset S_k \cdot \mathcal{H}(Z)$ for all $k \geq 0$. In this case the family φ_t , $t \in \mathbb{C}^*$, extends to a holomorphic family φ_t , $t \in \mathbb{C}$.

An automorphism $\varphi \in \text{Aut}(Z)$ acts on $\mathcal{D}^k(Z)$ sending P to $\varphi_*(P) := (\varphi^{-1})^* \circ P \circ \varphi^*$. If $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$, then φ acts on the holomorphic differential operators $\mathcal{H}\mathcal{D}^k(Z)$ of order k on Z by the same formula. If V is admissible, we have a surjection

$$\mathcal{H}\mathcal{D}^k(V)^G = \mathcal{H}(V)^G \otimes_{\mathcal{O}(V)^G} \mathcal{D}^k(V)^G \rightarrow \mathcal{H}(Z) \otimes_{\mathcal{O}(Z)} \mathcal{D}^k(Z) = \mathcal{H}\mathcal{D}^k(Z).$$

Theorem 2.3. *Let V be admissible and let $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$. Then φ is deformable.*

Proof. We may assume that $V^G = 0$. Let $f \in S_k$ and suppose that $\varphi^* f = h_\ell + F_{\ell+1} \mathcal{H}(Z)$ where $\ell < k$, $0 \neq h_\ell \in S_\ell$. Let $P \in \mathcal{O}(V^*)^G$ be dual to $\pi^* h_\ell$ under a choice of basis for V (and hence of $\mathcal{O}(V)$ and $\mathcal{O}(V^*)$). We may consider P as an invariant constant coefficient differential operator of order ℓ . Then $P(\pi^* h_\ell)$ is a nonzero constant and we can arrange that $P(\pi^* h_\ell) = 1$. Then $\varphi^*((\varphi_* \pi_* P)(f)) = (\pi_* P)(\varphi^* f)$ and $(\pi_* P)(\varphi^* f) = 1$ at $\pi(0)$, by construction. Now $\varphi_* \pi_* P$ has order at most ℓ (actually ℓ by [Sch95, Corollary 5.10]), hence it equals $\pi_* Q$ where $Q = \sum h_i Q_i$ and the h_i are in $\mathcal{H}(V)^G$ and the Q_i are in $\mathcal{D}^\ell(V)^G$. But each Q_i is a sum of terms of degree at least $-\ell$. Hence $\pi^*(\varphi_* \pi_* P(f)) = \sum_i h_i Q_i(\pi^*(f))$ where $Q_i(\pi^*(f))$ is a sum of terms of degree at least $k - \ell > 0$. Thus $Q_i(\pi^*(f))$ vanishes at 0 and $(\varphi_* \pi_* P)(f)$ vanishes at $\pi(0)$ which is a contradiction. Thus φ is deformable. \square

Corollary 2.4. *Let $G_i \subset \text{GL}(V_i)$, $i = 1, 2$. Assume that the V_i are admissible. Set $Z_i = V_i // G_i$ and suppose that there is a holomorphic isomorphism $\psi: Z_1 \rightarrow Z_2$. Then there is an algebraic isomorphism preserving the gradings of $\mathcal{O}(Z_1)$ and $\mathcal{O}(Z_2)$.*

Proof. Let $V = V_1 \oplus V_2$ and $G = G_1 \times G_2$. Then $Z := V // G \simeq Z_1 \times Z_2$. Let $\varphi: Z \rightarrow Z$ send (z_1, z_2) to $(\psi^{-1}(z_2), \psi(z_1))$ for $z_1 \in Z_1$, $z_2 \in Z_2$. Then φ is deformable, hence there is a graded isomorphism of $\mathcal{O}(Z_1)$ and $\mathcal{O}(Z_2)$. \square

Example 2.5. Let $(V_1, G_1) := (2n\mathbb{C}^{n-1}, \text{SL}_{n-1})$ and $(V_2, G_2) := (2n\mathbb{C}^{n+1}, \text{SL}_{n+1})$ where $n \geq 4$. The quotient Z_1 is isomorphic to the decomposable elements in $\wedge^{n-1}(\mathbb{C}^{2n})$ while Z_2 is isomorphic to the decomposable elements of $\wedge^{n+1}(\mathbb{C}^{2n})$. We have a canonical isomorphism $Z_1 \simeq Z_2$. The generators of $\mathcal{O}(V_1)^{G_1}$ have degree $n - 1$ while those of $\mathcal{O}(V_2)^{G_2}$ have degree $n + 1$. The representation V_1 is admissible while V_2 is not (but almost!). Both representations are 4-principal. Now consider $(V, G) = (V_1 \oplus V_2, G_1 \times G_2)$. Then the quotient is $Z_1 \times Z_2$ and we have an automorphism φ which interchanges the Z_i . It clearly is not deformable because of the difference in degrees of the generators. Thus Corollary 2.4 can fail when one of the representations is not admissible. Also, Theorem 1.11 and Corollary 1.12 can fail when \mathfrak{g}_s is not a subrepresentation of V .

Proposition 2.6. *Let $\varphi \in \text{Aut}_{\text{ql}}(Z)$ and suppose that φ lifts to $\Phi \in \text{End}(V)$ where V is 2-principal and φ is strata preserving (e.g., V is admissible or orthogonal). Then*

- (1) *There is a morphism $\sigma: G \rightarrow G$ such $\Phi(gv) = \sigma(g)\Phi(v)$, $g \in G$, $v \in V$.*
- (2) *$\Phi \in N_{\text{GL}(V)}(G)$, the normalizer of G in $\text{GL}(V)$.*
- (3) *$\sigma(g) = \Phi \circ g \circ \Phi^{-1}$, $g \in G$, so that $\sigma \in \text{Aut}(G)$.*
- (4) *If φ is the identity, then Φ is multiplication by an element $g \in G$.*

Proof. We know that φ preserves the principal stratum of Z and Φ sends the fiber of π over $z \in Z$ to the fiber over $\varphi(z)$. Consider a fiber which is a principal orbit Gv . Then Φ sends Gv to the principal orbit $G\Phi(v)$. Thus $\Phi(gv) = \Psi(v, g)\Phi(v)$ where $\Psi(v, g) \in G$ is unique. The morphism $\Psi: V_{\text{pr}} \times G \rightarrow G$ extends to a morphism of $V \times G$ to G since the complement of

V_{pr} has codimension at least two. By construction, $\Psi(v, g) = \Psi(tv, g)$ for any $t \in \mathbb{C}^*$. Hence $\Psi(v, g) = \sigma(g) := \Psi(0, g)$ for all principal v . It follows that $\Phi(gv) = \sigma(g)\Phi(v)$ for all $v \in V$ and we have (1).

It follows from (1) that $\text{Ker } \Phi$ is G -stable. Write $V = \text{Ker } \Phi \oplus V'$ where V' is G -stable. If $v_1 + v_2 \in \text{Ker } \Phi \oplus V'$, then $v_1 + v_2 \in V_{\text{pr}}$ if and only if $\Phi(v_2)$ is in V_{pr} . Thus every point of V_{pr} has a representative in V' and we get that $\dim V' // G = \dim V // G$. But $\dim V // G = \dim V - \dim G$ and $\dim V' // G \leq \dim V' - \dim G$. Hence $V' = V$ and $\Phi \in \text{GL}(V)$. Now for v principal we have that $\Phi(gv) = (\Phi \circ g \circ \Phi^{-1})\Phi(v)$, $g \in G$, so that $\sigma(g) = \Phi \circ g \circ \Phi^{-1}$. Hence we have (2) and (3).

Suppose that φ is the identity. Then for v principal there is a unique $\tau(v) \in G$ such that $\Phi(v) = \tau(v)v$. Arguing as above, τ extends to a morphism $V \rightarrow G$ and $\Phi(v) = \tau(0)v$ for all $v \in V$ and we have (4). \square

Remark 2.7. Let $v \in V$ and $g \in G$. We may change Φ to Φ_g where $\Phi_g(v) = \Phi(gv)$. Then for $h \in G$ we have

$$\Phi_g(hv) = \Phi(ghg^{-1}gv) = \sigma(g)\sigma(h)\sigma(g)^{-1}\Phi_g(v).$$

Thus we change σ by an inner automorphism of G . Hence if σ is inner, we can arrange that Φ is G -equivariant. In general, we can arrange that σ is a diagram automorphism (if G is semisimple).

Corollary 2.8. *Let V be admissible or 2-principal and orthogonal. Then V has the lifting property if and only if $N_{\text{GL}(V)}(G)$ maps onto $\text{Aut}_{\text{qel}}(Z)$.*

Proposition 2.9. *Let V be admissible or 2-principal and orthogonal. Then $\text{GL}(V)^G$ maps onto $\text{Aut}_{\text{qel}}(Z)^0$.*

Proof. Under either hypothesis, $\pi_*: \mathcal{D}^1(V)^G \rightarrow \mathcal{D}^1(Z)$ is surjective [Sch80, Theorem 0.2, Proposition 3.5]. Since π_* preserves the degrees of differential operators it sends the Lie algebra of $\text{GL}(V)^G$ which is $\text{End}(V)^G$ (the degree zero derivations) onto the degree zero derivations of $\mathcal{O}(Z)$ which are the Lie algebra of $\text{Aut}_{\text{qel}}(Z)$. \square

3. LIFTING HOLOMORPHIC ISOTOPIES

Now we have to show that we have lifting in case φ is deformable and $\varphi_0 := \lim_{t \rightarrow 0} t \circ \varphi \circ t$ lifts. Clearly we are reduced to the case that φ_0 is the identity. We use results of Heinzner [Hei91] and Heinzner and Kutzschebauch [HK95] to obtain an equivariant holomorphic lift of $\varphi_0^{-1}\varphi$.

Let X and Y be complex analytic G -varieties. Let K be a maximal compact subgroup of G . Let $\mathcal{H}(X, Y)$ denote the holomorphic maps from X to Y and let $\mathcal{H}(X, Y)^G$ denote the G -invariant ones. Similarly, we have $\mathcal{H}(X, Y)^K$. Since K is Zariski dense in G we have the following.

Lemma 3.1. $\mathcal{H}(X, Y)^G = \mathcal{H}(X, Y)^K$.

Let U be a K -stable open subset of X . We say that U is *orbit convex* if for every $Z \in i\mathfrak{k}$ and $x \in U$, if $\exp(Z)x \in U$, then $\exp(tZ)x \in U$ for $0 \leq t \leq 1$.

Proposition 3.2 ([Hei91]). *Let $U \subset V$ be K -invariant and orbit convex where V is a G -module. Let Y be a complex analytic G -variety. Then restriction gives an isomorphism of $\mathcal{H}(GU, Y)^G$ with $\mathcal{H}(U, Y)^K$.*

Let V be a G -module. Choose a K -invariant hermitian inner product on V . Then the *Kempf-Ness set* \mathcal{M} of V consists of the points $v \in V$ such that $T_v(Gv)$ is perpendicular to v . This set possesses some remarkable properties (see [Sch89]). In particular, $\mathcal{M} \rightarrow Z$ is surjective, proper and the fibers are K -orbits. Every closed orbit Gv has a closest point to the origin which is necessarily in \mathcal{M} .

Lemma 3.3 ([HK95]). *Let V be a G -module. Then any K -stable subset of \mathcal{M} has a neighborhood basis of orbit convex subsets U such that GU is G -saturated.*

The following result is a holomorphic analogue of the isotopy lifting theorem of [Sch80].

Theorem 3.4. *Let V be admissible and let $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$ such that φ_0 is the identity. Then there is a holomorphic equivariant family $\Phi_t: V \rightarrow V$, $t \in [0, 1]$, such that Φ_t lifts φ_t for each $t \in [0, 1]$.*

Proof. We have the deformation $\varphi_t = t^{-1} \circ \varphi \circ t$ for $t \in \mathbb{C}$. Differentiating in t we get a holomorphic (complex) time dependent vector field $A(t, z)$ such that the integral of A from 0 to t gives back φ_t . Since V is admissible, there is a holomorphic G -invariant vector field B on $\mathbb{C} \times V$ which covers A . We show that B can be integrated from time 0 to time 1. The integral Φ_t of B from time 0 to time t then clearly lifts φ_t , $t \in [0, 1]$.

Let $v_0 \in V$ such that Gv_0 is closed. We may assume that $v_0 \in \mathcal{M}$. Let B_a denote $(t, v) \mapsto B(a + t, v)$, $a \in [0, 1]$. Let L denote the inverse image in \mathcal{M} of $\varphi_t(\pi(v_0))$, $t \in [0, 1]$. Then L is compact and we can integrate B_a on L for $t \in [0, 2\epsilon]$ for some $\epsilon > 0$ and any $a \in [0, 1]$. Choose an orbit convex neighborhood U of L small enough so that we can integrate any B_a along $[0, \epsilon]$ for $v \in U$ and such that GU is G -saturated. Then by Proposition 3.2 we can extend the corresponding flow Φ_t^a of B_a to $[0, \epsilon] \times GU$. Note that Φ_t^a covers $\varphi_{t+a} \circ \varphi_a^{-1}$. Let $1/n < \epsilon$. Then $v_t := \Phi_t^0(v_0)$ covers $\varphi_t(\pi(v_0))$, $t \in [0, 1/n]$ and it lies in GU since GU is G -saturated. Thus we can apply $\Phi_t^{1/n}$ to $v_{1/n}$ for $t \in [0, 1/n]$ and we end up with $v_{2/n} \in GU$. Thus the flow Φ_t of B exists on a neighborhood of v_0 for $t \in [0, 2/n]$. Continuing inductively we see that the flow Φ_t exists in a neighborhood of v_0 for $t \in [0, 1]$. But v_0 is arbitrary in \mathcal{M} so that Φ_t , $t \in [0, 1]$ exists in an orbit convex neighborhood U of \mathcal{M} such that GU is G -saturated. But then $GU = V$. \square

Proof of Theorem 1.3. We just need to combine Theorems 2.3 and 3.4 and Propositions 2.1, 2.6 and 2.9. \square

4. SOME EXAMPLES

Lemma 4.1. *Let V be a representation of SL_n , $n \geq 3$, and let $\Phi \in \text{GL}(V)$ such that Φ normalizes the image H of SL_n in $\text{GL}(V)$. If V is not self-dual, then Φ induces an inner automorphism of H .*

Proof. We may assume that Φ is a diagram automorphism. If Φ induces an outer automorphism, then it sends every highest weight vector of V to a highest weight of V^* . Since V is not self-dual, the highest weights of V^* are not all weights of V . Hence Φ must induce an inner automorphism of H . \square

Here are the examples which arise in Theorem 1.8 and Remark 1.9.

Example 4.2. (See [Sch87] and [Sch88, §5].) Let $n \geq 4$. Then $(V, G) = (n\mathbb{C}^7, \mathbf{G}_2)$ is admissible. Since $V \simeq (\mathbb{C}^n)^* \otimes \mathbb{C}^7$ has a GL_n -action commuting with the \mathbf{G}_2 -action, GL_n acts on the invariants. We have generating invariants α_{ij} , β_{ijk} and γ_{ijkl} whose spans transform by the GL_n -representations $\text{S}^2(\mathbb{C}^n)$, $\wedge^3(\mathbb{C}^n)$ and $\wedge^4(\mathbb{C}^n)$, respectively. We use the notation ψ_q^p or $\psi_q^p(\mathbb{C}^n)$ to denote the Cartan component of $\text{S}^p(\wedge^q(\mathbb{C}^n))$.

There are three ways to get a highest weight element of $\psi_4^2(\mathbb{C}^n)$. There is $(\gamma_{1234})^2$, the determinant $\det(\alpha_{ij})_{i,j=1}^4$ and

$$\sum_{i,j=1}^4 (-1)^{i+j} \alpha_{ij} \hat{\beta}_i \hat{\beta}_j.$$

Here $\hat{\beta}_i = \beta_{klm}$ where $\{i, k, l, m\} = \{1, 2, 3, 4\}$ and $k < l < m$. We will denote these elements as $\lambda(\psi_4^2(\gamma^2))$, $\lambda(\psi_4^2(\alpha^4))$ and $\lambda(\psi_4^2(\alpha\beta^2))$, respectively. The notation is meant to denote a highest weight element of ψ_4^2 which is of the indicated degrees in the invariants.

From [Sch88, §5] there is a relation

$$(4.2.1) \quad \lambda(\psi_4^2(\gamma^2)) + \lambda(\psi_4^2(\alpha\beta^2)) - \lambda(\psi_4^2(\alpha^4)) = 0.$$

Since this is a relation of highest weight vectors, we actually have a whole GL_n -representation of relations. Now assume that $n = 4$. Then the relations are generated by the one above (we have a hypersurface). Let φ act on $\mathcal{O}(Z)$ such that it fixes the α_{ij} and β_{ijk} and sends γ_{1234} to its negative. If φ lifts to a linear Φ , then we can assume that Φ centralizes \mathbf{G}_2 since \mathbf{G}_2 has no outer automorphisms. Hence $\Phi \in \mathrm{GL}(V)^G = \mathrm{GL}_4$. But Φ acts trivially on $\mathbf{S}^2(\mathbb{C}^4)$, hence it is plus or minus the identity. Then Φ fixes γ_{1234} , showing that φ does not lift.

Suppose that $n > 4$. Then we pick up more relations. For example, we have a relation

$$(4.2.2) \quad \lambda(\psi_1\psi_5(\beta^2)) + \lambda(\psi_1\psi_5(\alpha\gamma)) = 0.$$

Let $\varphi \in \mathrm{Aut}_{\mathrm{qel}}(Z)$. The action of φ on $\mathbf{S}^2(\mathbb{C}^n)$ has to normalize the action of SL_n since SL_n times the scalars is $\mathrm{Aut}_{\mathrm{qel}}(Z)^0$. By Lemma 4.1 φ induces an inner automorphism of SL_n . Thus, modulo the image of an element of SL_n acting on $n\mathbb{C}^7$, we can assume that φ centralizes the action of SL_n , hence φ acts as a scalar on each of the types of invariants. Changing φ by the image of a scalar in GL_n , we can assume that φ acts trivially on $\mathbf{S}^2(\mathbb{C}^n)$. The relation (4.2.1) then shows that φ fixes each β_{ijk} or sends each β_{ijk} to its negative. We can cover this by choosing $-I \in \mathrm{GL}_n$, so we may assume that φ fixes the β_{ijk} . Then (4.2.2) shows that φ acts trivially on the γ_{ijkl} . Hence we have a lift of φ .

Example 4.3. (See [Sch88, §5].) Let $(V, G) = (n\mathbb{C}^8, \mathbf{B}_3)$. Then V is admissible for $n \geq 5$. The invariants are generated by inner products δ_{ij} and skew symmetric functions ϵ_{ijkl} where $\epsilon_{1234} \in \wedge^4(\mathbb{C}^8)^{\mathbf{B}_3} \subset \mathbf{S}^4(4\mathbb{C}^8)^{\mathbf{B}_3}$. For $n = 5$ we have a single generating relation

$$(4.3.1) \quad \lambda(\psi_5^2(\delta^5)) = \lambda(\psi_5^2(\delta\epsilon^2)).$$

As in the \mathbf{G}_2 case, the automorphism of Z which fixes the δ_{ij} and sends each ϵ_{ijkl} to its negative does not lift. Suppose that $n \geq 6$. Then we pick up the relation

$$(4.3.2) \quad \lambda(\psi_2\psi_6(\epsilon^2)) + \lambda(\psi_2\psi_6(\delta^2\epsilon)) = 0.$$

As in the \mathbf{G}_2 case, the relations allow one to show that any φ lifts.

Example 4.4. Let $(V, G) = (2n\mathbb{C}^n, \mathrm{SL}_n)$, $n \geq 2$. Then V is admissible. The generators are determinants of n vectors and transform by the representation $W := \wedge^n(\mathbb{C}^{2n})$ of GL_{2n} . There is a natural SL_{2n} -invariant bilinear form $\langle \cdot, \cdot \rangle$ on W which sends α, β to $\alpha \wedge \beta \in \wedge^{2n}(\mathbb{C}^{2n}) \simeq \mathbb{C}$. The bilinear form corresponds to an isomorphism ψ of W with W^* where $\psi(\alpha)(\beta) = \langle \alpha, \beta \rangle$. For the standard basis e_i for \mathbb{C}^{2n} let $\tau: \mathbb{C}^{2n} \rightarrow (\mathbb{C}^{2n})^*$ send e_i to e_i^* for each i . Recalling that $g \in \mathrm{SL}_{2n}$ acts on $(\mathbb{C}^{2n})^*$ by inverse transpose we have that $\tau(gv) = \sigma(g)\tau(v)$ for $v \in \mathbb{C}^{2n}$ and $g \in \mathrm{SL}_{2n}$ where $\sigma(g) = (g^{-1})^t$. We have an induced mapping (also called τ) from $W = \wedge^n(\mathbb{C}^{2n})$ to $W^* = \wedge^n((\mathbb{C}^{2n})^*)$ and composing with ψ^{-1} we have an isomorphism φ of W which is σ -equivariant. It is easy to see that φ preserves Z which is the set of decomposable vectors of W . If φ lifts, then it lifts to a linear mapping Φ of $(\mathbb{C}^{2n})^* \otimes \mathbb{C}^n$ which normalizes the SL_n -action and the SL_{2n} -action. By Lemma 4.1 the automorphism of SL_{2n} induced by Φ is inner. Thus Φ cannot cover φ , which acts on SL_{2n} via σ .

Example 4.5. Let $(V, G) = (k\mathbb{C}^n, \mathrm{SO}_n)$, $k \geq n \geq 3$. Then V is admissible. We have inner product invariants α_{ij} whose span transforms as $\mathbf{S}^2(\mathbb{C}^k)$ under the action of GL_k and determinant invariants β_{i_1, \dots, i_n} whose span transforms as $\wedge^n(\mathbb{C}^k)$. Let φ be quasilinear. As before, we can assume that the action of φ centralizes that of SL_k so that φ acts on the invariants α_{ij} and

β_{i_1, \dots, i_n} by scalars. We may reduce to the case that φ acts as the identity on $S^2(\mathbb{C}^k)$. We have the highest weight relation

$$(4.5.1) \quad \lambda(\psi_n^2(\alpha^n)) - \lambda(\psi_n^2(\beta^2)) = 0.$$

Now (4.5.1) implies that φ acts on the β_{i_1, \dots, i_n} as ± 1 . Consider an element of $O_n \setminus SO_n$. It fixes the α_{ij} and sends the β_{i_1, \dots, i_n} to their negatives. Hence φ lifts.

Example 4.6. Let $(V, G) = (k\mathbb{C}^n, O_n)$ where $k > n$. Then V is admissible. We only have generators α_{ij} and arguing as above one sees that $\text{Aut}_{\text{qel}}(Z)$ is the image of GL_k .

Example 4.7. Let $(V, G) = (k\mathbb{C}^{2n}, \text{Sp}_{2n})$, $k \geq 2n + 2$ and $n \geq 2$. Then V is admissible. The generators α_{ij} are skew in i and j and transform by the representation $\wedge^2(\mathbb{C}^k)$ of GL_k . Since $k \geq 6$ one can show as above that $\text{Aut}_{\text{qel}}(Z)$ is the image of GL_k .

Example 4.8. Let $(V, G) = (k\mathbb{C}^n + \ell(\mathbb{C}^n)^*, \text{SL}_n)$ where $k + \ell \geq 2n$ and $k + \ell > 2n$ if $k\ell = 0$. Then V is admissible. We leave the cases where $k\ell = 0$ or $k < n$ or $\ell < n$ to the reader and now consider the case where $k, \ell \geq n$. We use symbols ψ and $\bar{\psi}$ to differentiate between the representations of GL_k and GL_ℓ . Now $\text{GL}_k \times \text{GL}_\ell$ acts on the quotient $Z \subset \mathbb{C}^k \otimes \mathbb{C}^\ell \oplus \wedge^n(\mathbb{C}^k) \oplus \wedge^n(\mathbb{C}^\ell)$ where $\mathcal{O}(V)^G$ has corresponding generators α_{ij} (contractions), β_{i_1, \dots, i_n} (determinants of elements of $k\mathbb{C}^n$) and γ_{j_1, \dots, j_n} (determinants of elements of $\ell(\mathbb{C}^n)^*$). If $k \neq \ell$ then by Lemma 4.1 any $\varphi \in \text{Aut}_{\text{qel}}(Z)$ induces an inner automorphism of $\text{SL}_k \times \text{SL}_\ell$. If $k = \ell$ we have a φ which interchanges the copies of $\text{SL}_k = \text{SL}_\ell$. But this automorphism obviously lifts. Thus we may assume that φ induces an inner automorphism and changing φ by an element of $\text{SL}_k \times \text{SL}_\ell$ we may assume that φ commutes with $\text{GL}_k \times \text{GL}_\ell$. Then φ acts as a scalar on our three representations spaces. We have a relation

$$(4.8.1) \quad \lambda(\psi_n \bar{\psi}_n(\beta\gamma)) - \lambda(\psi_n \bar{\psi}_n(\alpha^n)) = 0.$$

This shows that the scalars a, b and c on the α_{ij} , etc. satisfy the relation $a^n = bc$. Let b' and c' be n th roots of b and c . Let b' (resp. c') act by multiplication on \mathbb{C}^k (resp. \mathbb{C}^ℓ). Then we get the desired action on the invariants of type β and γ and we act by $a := b'c'$ on the α_{ij} where $a^n = bc$. But $b'c'$ is an arbitrary n th root of bc . Hence φ lifts.

Example 4.9. Let $(V, G) = (k\mathbb{C}^n + \ell(\mathbb{C}^n)^*, \text{GL}_n)$, $k, \ell > n$. Then the same ideas as for SL_n show that any $\varphi \in \text{Aut}_{\text{qel}}(Z)$ lifts.

We have handled all the cases considered in Theorem 1.8 and Remark 1.9. Without burdening the reader with any more calculations we state the following proposition without proof. We denote by R_j the SL_2 -module $S^j(\mathbb{C}^2)$.

Proposition 4.10. *Let V be a G -module where $G = \text{SL}_2$.*

- (1) *Suppose that Z is a hypersurface singularity. Then any $\varphi \in \text{Aut}_{\text{qel}}(Z)$ lifts.*
- (2) *Suppose that $V = 2R_3$ (the unique case where Z is a complete intersection defined by 2 equations). Then any $\varphi \in \text{Aut}_{\text{qel}}(Z)$ lifts.*

Problem 4.11. Does one always have lifting for irreducible representations of SL_2 ?

5. TORI

We begin with a lemma which is surely well-known.

Lemma 5.1. *Let S and T be tori and let $\varphi: S \rightarrow T$ be a morphism of varieties which sends $e \in S$ to $e \in T$. Then φ is a homomorphism of algebraic groups.*

Proof. Let $m = \text{rank } S$ and $n = \text{rank } T$. Then $\varphi = (\varphi_1, \dots, \varphi_n)$ where $\varphi_j: S \rightarrow \mathbb{C}^*$. Thus we may reduce to the case that $T = \mathbb{C}^*$. Write $S = S_1 \times \mathbb{C}^*$ where S_1 has rank $m - 1$. For fixed $s_1 \in S_1$, set $\psi(t) = \varphi(s_1, 1)^{-1} \varphi(s_1, t): \mathbb{C}^* \rightarrow \mathbb{C}^*$. Then ψ is a unit in $\mathcal{O}(\mathbb{C}^*)$ which takes the value 1 at 1. Thus $\psi(t) = t^k$ for some k , i.e., ψ is a character of \mathbb{C}^* . By [Hum75, Proposition 16.3], the character we get is independent of $s_1 \in S_1$. Hence $\varphi(s_1, t) = \varphi(s_1, 1)\psi(t)$. Now $\varphi(s_1, 1): S_1 \rightarrow \mathbb{C}^*$ and by induction it is a character of S_1 . Thus φ is a character of S . \square

Let V be a 2-principal representation of G . Let Z and Z_{pr} be as usual and let Y denote $V//G^0$. Then $Z = Y/H$ where $H := G/G^0$. Let Y_{pr} denote the inverse image of Z_{pr} in Y .

Lemma 5.2. *Let Y , etc. be as above. Then any $\varphi \in \text{Aut}(Z)$ has a lift to $\text{Aut}(Y)$.*

Proof. Since $Y_{\text{pr}} \rightarrow Z_{\text{pr}}$ is a finite cover, there are local holomorphic lifts of φ over small open subsets of Z_{pr} . Since $Y \setminus Y_{\text{pr}}$ has real codimension four in Y , Y_{pr} is simply connected and we can patch together the local lifts to give a global holomorphic lift Φ on Y_{pr} . By [LMP03, Lemma 5.1.1], $\Phi \in \text{Aut}(Y_{\text{pr}})$, and Φ extends to an element of $\text{Aut}(Y)$ covering φ . \square

Let G_s be the semisimple part of G^0 and S its connected center.

Proposition 5.3. *Let V be 2-principal and let $\varphi \in \text{Aut}(Z)$. Then φ lifts to an automorphism $\Phi \in \text{Aut}(Y)$ where $Y = V//G_s$.*

Proof. Lemma 5.2 allows us to reduce to the case that G is connected. We may assume that S is nontrivial. Possibly dividing out by a finite kernel we may assume that S acts effectively on Y , in which case the principal isotropy group is trivial. Let $Z_0 := Z \setminus Z_{\text{pr}}$, let Y_0 denote the inverse image of Z_0 in Y and set $Y_{\text{pr}} := Y \setminus Y_0$. Then $\pi: Y_{\text{pr}} \rightarrow Z_{\text{pr}}$ is a principal S -bundle and Y_0 and Z_0 have complex codimension two in Y and Z , respectively. Let D be an irreducible hypersurface in Y_{pr} and let \overline{D} be its closure in Y . Since Y is factorial, \overline{D} is the zero set of an irreducible element of $\mathcal{O}(Y)$. Hence the divisor class group of Y_{pr} is trivial which implies that $H^1(Y_{\text{pr}}, \mathcal{O}^*)$ is trivial.

Let $L \in H^1(Z_{\text{pr}}, \mathcal{O}^*)$ be a line bundle. Then the pull back $\pi^*L \in H^1(Y_{\text{pr}}, \mathcal{O}^*)$ is trivial. Thus we have an isomorphism $\psi = (\psi_1, \psi_2): \pi^*L \rightarrow Y_{\text{pr}} \times \mathbb{C}$. Since $\pi^*L = \{(y, \ell) \mid y \in Y_{\text{pr}}, \ell \in L_{\pi(y)}\}$, it is invariant under multiplication by S on the first factor. Let m_s denote multiplication by $s^{-1} \in S$ on the Y_{pr} factor of π^*L and define $F: S \times Y_{\text{pr}} \rightarrow \mathbb{C}^*$ by $F(s, y)(\lambda) = \psi_2(m_s \psi^{-1}(y, \lambda))$; $s \in S, y \in Y_{\text{pr}}, \lambda \in \mathbb{C}$. Then $F(e, y) = 1$ for all $y \in Y_{\text{pr}}$. By Lemma 5.1 and [Hum75, Proposition 16.3], $F(s, y) = \chi(s)$ where χ is a character of S which is independent of $y \in Y_{\text{pr}}$. Let \mathbb{C}_χ denote a copy of \mathbb{C} where S acts via χ . Then L is isomorphic to the line bundle $L_\chi := Y_{\text{pr}} \times^S \mathbb{C}_\chi$ and $H^1(Z_{\text{pr}}, \mathcal{O}^*) \simeq X(S)$, the character group of S .

Now φ^*L_χ is isomorphic to a line bundle $L_{\tau(\chi)}$ for some $\tau(\chi) \in X(S)$. Using additive notation for the group structure of $X(S)$ we see that $\tau(\chi_1 + \chi_2) = \tau(\chi_1) + \tau(\chi_2)$ for $\chi_1, \chi_2 \in X(S)$. Thus φ acts on $H^1(Z_{\text{pr}}, \mathcal{O}^*) \simeq X(S)$ by $\tau \in \text{GL}_n(\mathbb{Z})$ where n is the rank of S . Let χ_1, \dots, χ_n be a basis of $X(S)$. For $j = 1, \dots, n$ we have an isomorphism ψ_j of $\varphi^*L_{\chi_j}$ with $L_{\tau(\chi_j)}$. From ψ_j we canonically obtain an isomorphism denoted ψ_j^{-1} of $\varphi^*L_{-\chi_j} = (\varphi^*L_{\chi_j})^{-1}$ with $L_{-\tau(\chi_j)}$. Let $\chi = \sum n_j \chi_j$. Then define

$$\psi_\chi = \bigotimes_{j=1}^n \psi_j^{n_j}: (\varphi^*L_\chi = \bigotimes_{j=1}^n (\varphi^*L_{\chi_j})^{n_j}) \rightarrow (L_{\tau(\chi)} = \bigotimes_{j=1}^n (L_{\tau(\chi_j)})^{n_j}).$$

Then the isomorphisms ψ_χ give us an algebra isomorphism

$$\psi: \bigoplus_{\chi \in X(S)} \Gamma(Z_{\text{pr}}, \varphi^*L_\chi) \simeq \bigoplus_{\chi \in X(S)} \Gamma(Z_{\text{pr}}, L_{\tau(\chi)}) = \bigoplus_{\chi \in X(S)} \Gamma(Z_{\text{pr}}, L_\chi).$$

Let $\mathcal{O}(Y)_\chi$ denote the covariants of type χ , that is, $f \in \mathcal{O}(Y)_\chi$ if $f(s^{-1}y) = \chi(s)f(y)$ for $y \in Y$ and $s \in S$. Then

$$\mathcal{O}(Y) = \bigoplus_{\chi \in X(S)} \mathcal{O}(Y)_\chi = \bigoplus_{\chi \in X(S)} \mathcal{O}(Y_{\text{pr}})_\chi = \bigoplus_{\chi \in X(S)} \Gamma(Z_{\text{pr}}, L_\chi).$$

Let $L_\chi(z)$ denote the fiber of L_χ at z . Then for $z \in Z_{\text{pr}}$ we have

$$\mathcal{O}(\pi^{-1}(z)) \simeq \bigoplus_{\chi \in X(S)} L_\chi(z) \text{ and } \mathcal{O}(\pi^{-1}(\varphi(z))) \simeq \bigoplus_{\chi \in X(S)} (\varphi^* L_\chi)(z).$$

Thus ψ gives us a family of isomorphisms of the fibers $\pi^{-1}(z)$ and $\pi^{-1}(\varphi(z))$ parameterized by Z_{pr} . Thus we have a lift Φ of φ to Y_{pr} and Φ canonically extends to a lift of φ to Y . By construction, Φ is an automorphism. \square

Proof of Theorem 1.10. Let $\varphi \in \text{Aut}(Z)$. Then we have constructed $\Phi \in \text{Aut}(V)$ such that $\Phi^*(\mathcal{O}(V)_\chi) = \mathcal{O}(V)_{\tau(\chi)}$ for all $\chi \in X(S)$. Let $f \in \mathcal{O}(V)_\chi$, $s \in S$, $v \in V_{\text{pr}}$. Then $(\Phi^* f)(s^{-1}v) = \tau(\chi)(s)f(\Phi(v))$. Since $\Phi(s^{-1}v) \in S\Phi(v)$, we are forced to have $\Phi(s^{-1}v) = \sigma(s^{-1})\Phi(v)$ where $\chi \circ \sigma = \tau(\chi)$. Thus σ is an automorphism of S and Φ is σ -equivariant. If we start out with $\varphi \in \text{Aut}_{\mathcal{H}}(Z)$, then we know that φ is deformable to a $\varphi_0 \in \text{Aut}_{\text{ql}}(Z)$, which, by Proposition 5.3 and our reasoning above, lifts to a σ -equivariant automorphism of V . Then by Theorem 1.3, φ has a σ -equivariant lift to $\text{Aut}_{\mathcal{H}}(V)$. \square

Later we will need a local version of lifting.

Proposition 5.4. *Assume that V is a 2-principal G -module where G^0 is a torus. Let B and B' be neighborhoods of $z_0 := \pi(0)$ and let $\varphi: B \rightarrow B'$ be biholomorphic where $\varphi(z_0) = z_0$. Then, after perhaps shrinking B and B' , we can find a σ -equivariant holomorphic lift Φ of φ .*

Proof. Using the argument of Lemma 5.2 we can reduce to the case that G is connected. Define φ_t as usual. Then, since V is admissible, the proof of Theorem 2.3 shows that $\varphi_0 := \lim_{t \rightarrow 0} \varphi_t$ exists. Since all the φ_t preserve z_0 , we can shrink B so that $\varphi_t(B) \subset B'$, $t \in [0, 1]$. By Proposition 5.3, φ_0 lifts, hence we can reduce to the case that φ_0 is the identity. Then the argument of Theorem 3.4 shows that we have an equivariant lift Φ_t of φ_t , $0 \leq t \leq 1$. Replacing B' by $\varphi(B)$ we see that Φ_1 is the required lift of φ . \square

6. THE METHOD OF KUTTLE

We give a proof of Theorem 1.11. Let V be a 4-principal G -module which we may assume is faithful. Let V_0 denote $V \setminus V_{\text{pr}}$. By Proposition 5.3, lifting for elements of $\text{Aut}(Z)$ reduces to establishing lifting for the action of G_s where G_s is the semisimple part of G^0 . Since (V, G) is 4-principal, it is easy to see that (V, G_s) is also 4-principal. Hence we may assume that G is connected semisimple. In Theorem 1.11 we assume that

(*) $V = \mathfrak{g} \oplus V'$, i.e., V contains a copy of the adjoint representation of G .

Lemma 6.1. *The mapping $G \rightarrow \text{GL}(V')$ is injective.*

Proof. Let K be the kernel, assumed to be nontrivial. If $\dim K > 0$, then K contains a simple component of G whose representation on \mathfrak{g} has codimension one strata. Hence V is not 2-principal and we have a contradiction. Thus K is finite and central and acts trivially on \mathfrak{g} , contradicting the fact that $G \rightarrow \text{GL}(V)$ is faithful. Hence K is trivial. \square

Let $\mathfrak{A}(Z)$ denote the vector fields on Z (derivations of $\mathcal{O}(Z)$) and define $\mathfrak{A}(V)$ similarly. Since the elements of \mathfrak{g} act on V as vector fields, we have an inclusion $\tau: \mathfrak{g} \rightarrow \mathcal{O}(V) \otimes V \simeq \mathfrak{A}(V)$. Thus we have an injection (also called τ) from $M_{\mathfrak{g}} := \mathcal{O}(V) \otimes \mathfrak{g}$ to $M_V := \mathcal{O}(V) \otimes V$. Let M denote $M_V/M_{\mathfrak{g}}$.

Lemma 6.2. *The module M has depth 3 relative to the ideal of V_0 .*

Proof. The free $\mathcal{O}(V)$ -modules $M_{\mathfrak{g}}$ and M_V have depth 4 with respect to the ideal of V_0 , hence M has depth at least 3. \square

Set $E := (M_{\mathfrak{g}})^G$ and $F := (M_V)^G = \mathfrak{A}(V)^G$. Then $\tau(E) \subset F$. The image of τ consists of invariant vector fields tangent to the G -orbits. We have a morphism $\pi_*: \mathfrak{A}(V)^G \rightarrow \mathfrak{A}(Z)$ which just restricts an element of $\mathfrak{A}(V)^G$ to $\mathcal{O}(V)^G = \mathcal{O}(Z)$. By [Sch95, Theorem 8.9] we have that

$$(**) \quad 0 \rightarrow E \xrightarrow{\tau} F \xrightarrow{\pi_*} \mathfrak{A}(Z) \rightarrow 0 \text{ is exact.}$$

Let \mathcal{E} , \mathcal{F} and \mathcal{G} be the sheaves of \mathcal{O}_Z -modules corresponding to E , F and $\mathfrak{A}(Z)$, respectively. Let \mathcal{F}' be the sheaf corresponding to $(\mathcal{O}(V) \otimes V')^G$, so we have that $\mathcal{F} \simeq \mathcal{E} \oplus \mathcal{F}'$. The sections of \mathcal{E} over Z_{pr} are the sections of the vector bundle $V_{\text{pr}} \times^G \mathfrak{g}$ so that \mathcal{E} is locally free on Z_{pr} . Similarly, \mathcal{F} is locally free on Z_{pr} and the quotient \mathcal{G} is locally free as well.

Let L be a finite $\mathcal{O}(Z)$ -module and let \mathcal{L} denote the corresponding sheaf on Z , which we assume to be locally free on Z_{pr} . Let $\pi^*\mathcal{L}$ denote $(\pi|_{V_{\text{pr}}})^*\mathcal{L}$. Then $\pi^*\mathcal{L}$ is locally free on V_{pr} , hence reflexive, and $N := \Gamma(V_{\text{pr}}, \pi^*\mathcal{L})$ is a finitely generated reflexive $\mathcal{O}(V)$ -module (see [Kut11, Remark 13]). Let $\pi^\#\mathcal{L}$ denote the coherent sheaf of \mathcal{O}_V -modules corresponding to N . For any open subset U of V , $\Gamma(U, \pi^\#\mathcal{L}) = \Gamma(U \cap V_{\text{pr}}, \pi^*\mathcal{L})$ so that $\pi^\#\mathcal{L} = i_*\pi^*\mathcal{L}$ where $i: V_{\text{pr}} \rightarrow V$ is inclusion.

Since $\pi|_{V_{\text{pr}}}$ is flat we have a complex of coherent sheaves

$$(6.2.1) \quad 0 \rightarrow \pi^\#\mathcal{E} \rightarrow \pi^\#\mathcal{F} \rightarrow \pi^\#\mathcal{G} \rightarrow 0$$

which is exact on V_{pr} . Since $V_{\text{pr}} \times^G \mathfrak{g}$ is locally trivial over Z_{pr} , $\pi^*\mathcal{E}$ is locally the sheaf of sections of $V_{\text{pr}} \times \mathfrak{g}$. Thus the global sections of $\pi^\#\mathcal{E}$ are a reflexive module over $\mathcal{O}(V_{\text{pr}}) = \mathcal{O}(V)$ which agrees with $M_{\mathfrak{g}}$ on V_{pr} , hence the global sections are $M_{\mathfrak{g}}$. Similarly, the global sections of $\pi^\#\mathcal{F}$ are M_V . The global sections of $\pi^\#\mathcal{G}$ are a reflexive module which agrees with the locally free module M over V_{pr} . Hence the global sections are the double dual M^{**} . Since M has depth 3 (2 will do) with respect to the ideal of V_0 we have that $M = M^{**}$ ([Sch95, Lemma 8.3]). Hence $\pi^\#\mathcal{G}$ corresponds to M and (6.2.1) is exact on V .

Let $\varphi \in \text{Aut}(Z)$. Then applying pull back by φ we obtain an exact sequence

$$(6.2.2) \quad 0 \rightarrow \varphi^*\mathcal{E} \rightarrow \varphi^*\mathcal{F} \xrightarrow{\pi_*} \varphi^*\mathcal{G} \rightarrow 0.$$

We alter the map π_* in (6.2.2) by composing it with $d\varphi^{-1}: T_{\varphi(z)}Z \rightarrow T_z(Z)$, $z \in Z$. On vector fields composing with $d\varphi^{-1}$ is the automorphism $A \mapsto (\varphi^{-1})_*A := \varphi^* \circ A \circ (\varphi^{-1})^*$, $A \in \mathfrak{A}(Z)$, and $(\varphi^{-1})_*$ gives an isomorphism of $\varphi^*\mathfrak{A}(Z)$ with $\mathfrak{A}(Z)$. Then we have a complex

$$(6.2.3) \quad 0 \rightarrow \pi^\#\varphi^*\mathcal{E} \rightarrow \pi^\#\varphi^*\mathcal{F} \rightarrow \pi^\#\mathcal{G} \rightarrow 0$$

which is exact on V_{pr} . The key idea, following [Kut11], is to show that $\Gamma(V_{\text{pr}}, \pi^\#\varphi^*\mathcal{F}') = \Gamma(V, \pi^\#\varphi^*\mathcal{F}')$ is a free $\mathcal{O}(V)$ -module.

Now we take cohomology over V_{pr} to get an exact sequence

$$0 \rightarrow \Gamma(V, \pi^\#\varphi^*\mathcal{E}) \rightarrow \Gamma(V, \pi^\#\varphi^*\mathcal{F}) \rightarrow M \rightarrow H^1(V_{\text{pr}}, \pi^\#\varphi^*\mathcal{E}) \rightarrow H^1(V_{\text{pr}}, \pi^\#\varphi^*\mathcal{F}) \rightarrow H^1(V_{\text{pr}}, \pi^\#\mathcal{G})$$

where the last module is zero since M has depth 3 with respect to the ideal of V_0 . The map $H^1(V_{\text{pr}}, \pi^\#\varphi^*\mathcal{E}) \rightarrow H^1(V_{\text{pr}}, \pi^\#\varphi^*\mathcal{F})$ is a surjective morphism of finitely generated $\mathcal{O}(V)$ -modules ([Kut11, Remark 13]). Since $\pi^\#\varphi^*\mathcal{F}$ contains $\pi^\#\varphi^*\mathcal{E}$ as a direct summand, localizing at points of V and applying Nakayama's lemma we see that $H^1(V_{\text{pr}}, \pi^\#\varphi^*\mathcal{E}) \rightarrow H^1(V_{\text{pr}}, \pi^\#\varphi^*\mathcal{F})$ has to be injective. Hence the map is an isomorphism and it follows that $\Gamma(V, \pi^\#\varphi^*\mathcal{F})$ maps onto M . Thus we have exact sequences of finite $\mathcal{O}(V)$ -modules

$$(6.2.4) \quad 0 \rightarrow M_{\mathfrak{g}} \rightarrow M_V \rightarrow M \rightarrow 0 \text{ and}$$

$$(6.2.5) \quad 0 \rightarrow \Gamma(V, \pi^\# \varphi^* \mathcal{E}) \rightarrow \Gamma(V, \pi^\# \varphi^* \mathcal{F}) \rightarrow M \rightarrow 0.$$

Let

$$\Psi: M_V \oplus \Gamma(V, \pi^\# \varphi^* \mathcal{F}) \rightarrow M \rightarrow 0$$

be the difference of the two surjections.

Lemma 6.3. *We have exact sequences*

$$(6.3.1) \quad 0 \rightarrow M_{\mathfrak{g}} \rightarrow \text{Ker } \Psi \rightarrow \Gamma(V, \pi^\# \varphi^* \mathcal{F}) \rightarrow 0 \text{ and}$$

$$(6.3.2) \quad 0 \rightarrow \Gamma(V, \pi^\# \varphi^* \mathcal{E}) \rightarrow \text{Ker } \Psi \rightarrow M_V \rightarrow 0.$$

Proof. In (6.3.1) we have $M_{\mathfrak{g}} \subset M_V \subset M_V \oplus \Gamma(V, \pi^\# \varphi^* \mathcal{F})$. Then exactness of (6.2.4) implies that $\text{Ker } \Psi / M_{\mathfrak{g}}$ maps isomorphically onto $\Gamma(V, \pi^\# \varphi^* \mathcal{F})$ via projection. The argument for (6.3.2) is similar. \square

Since M_V is a projective G and $\mathcal{O}(V)$ -module, we can find a G -equivariant $\mathcal{O}(V)$ -module splitting of (6.3.2). Thus we may replace $\text{Ker } \Psi$ in (6.3.1) by $\Gamma(V, \pi^\# \varphi^* \mathcal{E}) \oplus M_V$ and we obtain an exact sequence

$$0 \rightarrow M_{\mathfrak{g}} \rightarrow \Gamma(V, \pi^\# \varphi^* \mathcal{E}) \oplus M_{\mathfrak{g}} \oplus (\mathcal{O}(V) \otimes V') \rightarrow \Gamma(V, \pi^\# \varphi^* \mathcal{F}') \oplus \Gamma(V, \pi^\# \varphi^* \mathcal{E}) \rightarrow 0.$$

We obtain an exact sequence from the one above by tensoring over $\mathcal{O}(V)$ with $\mathcal{O}_{V,v} / \mathcal{M}_v \simeq \mathbb{C}$ where \mathcal{M}_v is the maximal ideal of the local ring $\mathcal{O}_{V,v}$, $v \in V$. We deduce that

$$\Gamma(V, \pi^\# \varphi^* \mathcal{F}') \otimes_{\mathcal{O}(V)} \mathcal{O}_{V,v} / \mathcal{M}_v \text{ and } (\mathcal{O}(V) \otimes V') \otimes_{\mathcal{O}(V)} \mathcal{O}_{V,v} / \mathcal{M}_v \simeq \mathbb{C}^{\dim V'}$$

have the same dimension. It follows that $\Gamma(V, \pi^\# \varphi^* \mathcal{F}')$ is a projective $\mathcal{O}(V)$ -module, hence free by Quillen and Suslin.

Now $P := V_{\text{pr}} \rightarrow Z_{\text{pr}}$ is a principal G -bundle as is $\varphi^* P$. The pull-back $P' := (\pi|_{V_{\text{pr}}})^* \varphi^* P$ is a principal G -bundle over V_{pr} . We have associated vector bundles $P \times^G V'$, $\varphi^* P \times^G V'$ and $P' \times^G V'$. It is standard that the vector bundle $P' \times^G V'$ is trivial if and only if the principal bundle $P' \times^G \text{GL}(V')$ is trivial. The sections of $P \times^G V'$ and \mathcal{F}' over Z_{pr} are the same, the sections of $\varphi^* P \times^G V'$ and $\varphi^* \mathcal{F}'$ over Z_{pr} are the same and the sections of $P' \times^G V'$ and $\pi^\# \varphi^* \mathcal{F}'$ over V_{pr} are the same. We have shown that $\Gamma(V, \pi^\# \varphi^* \mathcal{F}')$ is a free $\mathcal{O}(V)$ -module, hence $P' \times^G V'$ is a trivial vector bundle and $P' \times^G \text{GL}(V')$ is a trivial principal bundle. Now $G \rightarrow \text{GL}(V')$ is injective and it follows that P' is the pull-back of the principal G -bundle $\text{GL}(V') \rightarrow \text{GL}(V')/G$ via a morphism $V_{\text{pr}} \rightarrow \text{GL}(V')/G$. Since $\text{GL}(V')/G$ is affine the morphism extends to V . Hence P' extends to a principal G -bundle P'' over V . By [Rag78] P'' is trivial, hence so is P' . Now $P = V_{\text{pr}}$, so that $P' = P \times_{Z_{\text{pr}}} \varphi^* P \simeq P \times G$. Thus P' has a section. This is the same thing as a mapping Φ of P to $\varphi^* P$ which respects the fibers of the maps to Z_{pr} . Since $\varphi^* P$ has fiber $\pi^{-1}(\varphi(\pi(v)))$ at $\pi(v)$, Φ is a map of V_{pr} to V_{pr} which is a lift of φ . Then Φ extends to a morphism of V to V which lies over φ . This completes the proof of Theorem 1.11.

7. MULTIPLES OF THE ADJOINT REPRESENTATION

We give a proof of Theorem 1.14. Recall that $V = \bigoplus_{i=1}^d r_i \mathfrak{g}_i$ where $r_i \geq 2$ unless $\mathfrak{g}_i = \mathfrak{sl}_2$, in which case $r_i \geq 3$. Let G_i denote the adjoint group of \mathfrak{g}_i and set $G = G_1 \times \cdots \times G_d$. Let Z_i denote $(r_i \mathfrak{g}_i) // G_i$. Then $Z = Z_1 \times \cdots \times Z_d$. We cannot just apply Theorem 1.11 because $\pi^{-1}(V \setminus V_{\text{pr}})$ can have codimension less than four. If S is a stratum of Z with $\text{codim } S \geq 6$, then $\text{codim } \pi^{-1}(S) \geq 4$ so we have to worry about the strata of codimension at most 5. We classify them and show that we can reduce to the case that they are fixed by our automorphism φ of Z . The corresponding isotropy groups are tori or finite, so we can use Proposition 5.4 to find local holomorphic lifts of φ over these strata. This enables us to show that the sequence (6.2.3)

is exact on V^\dagger where V^\dagger is the inverse image of the strata of Z of codimension at most 5 (so V^\dagger is open). Since $\text{codim}(V \setminus V^\dagger) \geq 4$, the rest of the proof of Theorem 1.11 goes through.

The strata of Z are products $S_1 \times \cdots \times S_d$ where S_i is a stratum of Z_i . We will see that, unless S_i is the principal stratum, it has codimension at least three. Thus the strata of codimension at most five are the product of an S_i of codimension at most five with the principal strata of the other factors.

We begin with the case where $V = r\mathfrak{g}$, \mathfrak{g} is simple, and G is the adjoint group of \mathfrak{g} (i.e., $n = 1$). Let $S = Z_{(H)}$ be a stratum with associated slice representation (W, H) . We have an H -module decomposition $W = W^H \oplus W'$ where W' is an H -module. It is easy to calculate W since we have the equality of H -modules $V = W \oplus \mathfrak{g}/\mathfrak{h}$. Since $V = r\mathfrak{g}$ we get the formula

$$W = (r - 1)\mathfrak{g} \oplus \mathfrak{h}.$$

Note that W is orthogonal. It follows from the slice theorem that the codimension of S in Z is $\dim W' // H$ and that the codimension of $\pi^{-1}(S)$ is the codimension of the null cone $\mathcal{N}(W')$ in W' .

The case where $\mathfrak{g} = \mathfrak{sl}_2$ is trivial and is summed up in the following lemma, which we leave to the reader. For $H = \mathbb{C}^*$ we denote by ν_j the one-dimensional module of weight $j \in \mathbb{N}$. We let θ denote a trivial module of any dimension.

Lemma 7.1. *Let $V = r\mathfrak{sl}_2$ and $G = \text{PSL}_2$ where $r \geq 3$. Then there are two non principal strata as follows.*

- (1) $(W, H) = ((r - 1)(\nu_1 + \nu_{-1}) + \theta, \mathbb{C}^*)$. Thus $\text{codim } S = 2r - 3$ and $\text{codim } \pi^{-1}(S) = r - 1$.
- (2) $(W, H) = (V, G)$ so S is the origin of Z and has codimension $\dim Z = 3(r - 1)$.

For the next few results we assume that $\text{rank } \mathfrak{g} \geq 2$. Let S be a stratum with slice representation $(W = W^H \oplus W', H)$. If H is finite, the codimension of $\pi^{-1}(S)$ is $\dim W'$, else from [Sch80, Corollary 10.2] we have the estimate

$$\text{codim } \pi^{-1}(S) = \text{codim } \mathcal{N}(W') \geq \frac{1}{2}(\dim W' // H + \text{rank } H + \mu(W'))$$

where $\mu(W')$ is the multiplicity of the zero weight.

Lemma 7.2. *Let S be a stratum with slice representation (W, H) .*

- (1) *If $\text{codim } S \geq 6$, then $\text{codim } \pi^{-1}(S) \geq 4$.*
- (2) *If $\text{codim } S \leq 5$, then H^0 is a torus.*

Proof. For (1) note that $\text{codim } \pi^{-1}(S) \geq 6$ if H is finite, else $\text{rank } H \geq 1$ and our estimate above gives that $\text{codim } \pi^{-1}(S) \geq 7/2$, hence $\text{codim } \pi^{-1}(S) \geq 4$. For (2), suppose that H has semisimple rank at least two. Then the action of H on W' contains at least twice \mathfrak{h} which implies that $\dim W' // H \geq 6$. If H has semisimple rank 1, then W' , as SL_2 -module, contains more than $2\mathfrak{sl}_2$ since this representation has codimension one strata. Thus, at a minimum, we have an orthogonal representation of SL_2 times a torus T on a representation of the form $2\mathfrak{sl}_2 + R_{2m+1}$, $m \geq 1$, or of the form $2\mathfrak{sl}_2 + R_m \otimes \nu_\alpha + R_m \otimes \nu_{-\alpha}$, $m \geq 1$. Recall that $R_m = S^m(\mathbb{C}^2)$ and we denote by ν_α the one-dimensional representation where T acts by the character α . Thus we get a quotient of dimension at least 6. Hence H^0 is a torus and we have (2). \square

We call a stratum S *subprincipal* if its slice representation (W, H) is nontrivial and the closed orbits of (W, H) are either fixed points or principal. Then S is not in the closure of any stratum except the principal one. From [Kut11] we borrow the following.

Lemma 7.3. *Let $V = r\mathfrak{g}$ be as above and let T denote a maximal torus of G . Let S be a subprincipal stratum. Then H is conjugate to a subgroup of T which is finite or a one-dimensional torus.*

Corollary 7.4. *Suppose that $\text{codim } S \leq 5$ and that H is finite.*

- (1) *If S is subprincipal, then it has codimension 4.*
- (2) *If S is not subprincipal, it has codimension 5 and is contained in the closure of a subprincipal stratum S' of codimension four where the associate isotropy group H' is finite.*

Proof. For (1) we know that H is contained in a maximal torus T of G . The action of H on $(r-1)\mathfrak{g}$ (which is the slice representation up to trivial factors) acts on the root spaces \mathfrak{g}_α via the roots α applied to H (thought of as characters of T). Since the roots appear in pairs $\pm\alpha$, the dimension of W' is even, hence so is the codimension of S . If this codimension is two, then $r = 2$ and all the simple roots have value one on H , except for a pair $\pm\alpha$. But then there is an adjacent simple root β such that $\alpha + \beta$ is a root and $\alpha + \beta$ applied to H is nontrivial. Hence we have a contradiction, which gives us (1).

For (2) we know that S corresponds to a non principal slice representation of a finite group, hence there is a stratum S' of codimension less than 5 corresponding to a finite group. By (1) S' has codimension 4, hence we have (2). \square

Corollary 7.5. *Let $V = r\mathfrak{g}$ be as above. Then Z has no codimension two strata. For any codimension three stratum we have $(W, H) = (2\nu_1 + 2\nu_{-1} + \theta, \mathbb{C}^*)$.*

Proof. We know that Z has no codimension one strata [Sch13, Proposition 3.1]. If S is of codimension two, it is subprincipal. By Corollary 7.4 H cannot be finite, so we have that $H = \mathbb{C}^* \subset T$. As in the proof above, there are at least two pairs of roots $\pm\alpha, \pm\beta$ which are non trivial on H . Hence W has at least four nonzero weights and $\text{codim } S \geq 3$. Thus there are no codimension two strata. If S has codimension three, then it must be subprincipal, hence the isotropy group is \mathbb{C}^* . But a subprincipal orthogonal faithful representation of \mathbb{C}^* has to have weights in pairs ± 1 . \square

Lemma 7.6. *Let G be simple adjoint such that $(V := r\mathfrak{g}, G)$ has a stratum of codimension 3 (we allow $\text{rank } G = 1$). Then $(r\mathfrak{g}, G) = (2\mathfrak{sl}_3, \text{PSL}_3)$ or $(3\mathfrak{sl}_2, \text{PSL}_2)$.*

Proof. Lemma 7.1 takes care of the case where $G = \text{PSL}_2$, so assume that $\text{rank } \mathfrak{g} \geq 2$. We may assume that $H = \mathbb{C}^*$ acts with nonnegative weights on the positive root spaces. If there were two simple roots α and β which are nontrivial on H , then either $\alpha + \beta$ is a root and is nontrivial on H or $\alpha + \gamma$ is nontrivial where γ is a third simple root. Hence only one simple root α is nontrivial on H . Then the other root not vanishing on H is of the form $\alpha + \beta$ where β is adjacent to α . We clearly must have $r = 2$, else there are more than 4 nonzero weight spaces of H . Moreover, the root α must be adjacent to at most one other simple root β and $\alpha + \beta$ must be the only positive root of \mathfrak{g} for which α and β have positive coefficients. Hence the highest root is $\alpha + \beta$, i.e., $\mathfrak{g} = \mathfrak{sl}_3$. Finally, one only has to check that $2\mathfrak{sl}_3$ has the desired slice representation. \square

Remark 7.7. Examining the slice representations of $3\mathfrak{sl}_2$ and $2\mathfrak{sl}_3$ one sees the following where S denotes the codimension three stratum.

- (1) If $V = 3\mathfrak{sl}_2$, then S contains only a stratum of codimension 6 in its closure (the image of the fixed point).
- (2) If $V = 2\mathfrak{sl}_3$, then S contains a stratum of codimension 4 in its closure (the image of the points with isotropy group the maximal torus) and the next stratum has codimension 6 (corresponding to an isotropy group of semisimple rank 1). The closure of S is all the non principal strata of Z .

Corollary 7.8. *Suppose that $\text{codim } S = 4$. Then either*

- (1) *S is subprincipal and H is finite, or*
- (2) *H is the maximal torus of PSL_3 , $V = 2\mathfrak{sl}_3$ and S is not subprincipal.*

Proof. Suppose that S is not subprincipal. Then it has a codimension three stratum in its closure, and Remark 7.7 gives us the desired result. Thus S is subprincipal. If H is not finite, then $H = \mathbb{C}^*$ and W has an odd number of weights, which is impossible. Hence H is finite. \square

Corollary 7.9. *Suppose that S has codimension five. Then H has rank at most one.*

Proof. Suppose that H has rank $k > 1$. Let α be a character of the H^0 -action on W' with associated one-dimensional weight space ν_α . Then we have an H^0 -stable subrepresentation $\nu_\alpha + \nu_{-\alpha}$ of W' and there is a closed orbit in the subrepresentation with isotropy group of rank $k - 1$. This corresponds to a stratum of codimension 3 or 4. Codimension 3 cannot occur by Corollary 7.5, so we have a stratum of codimension four with isotropy group of rank $k - 1$. Then by Corollary 7.8 we have $k = 3$ and $V = 2\mathfrak{sl}_3$ which is impossible since PSL_3 has rank 2. Hence H has rank at most one. \square

Proposition 7.10. *Suppose that $\mathrm{codim} S$ is four or five and that H is finite. Then $\mathrm{codim} S = 4$, $V = 2\mathfrak{so}_5$ and $H = \pm 1$ acting by scalar multiplication on $W' = \mathbb{C}^4$. Suppose that S has codimension 5. Then $H = \mathbb{C}^*$ and we have the following possibilities.*

- (1) $V = 4\mathfrak{sl}_2$. Then S is subprincipal and the only stratum in its closure has codimension 9.
- (2) $V = 3\mathfrak{sl}_3$. Then S is subprincipal and the maximal stratum in the closure has codimension 10.
- (3) $V = 2\mathfrak{so}_5$ and $W' = 3\nu_1 + 3\nu_{-1}$. Then S is subprincipal and the maximal stratum in its closure has codimension 6.
- (4) $V = 2\mathfrak{so}_5$ and $W' = 2\nu_1 + 2\nu_{-1} + \nu_2 + \nu_{-2}$. Then S is not subprincipal, it lies in the closure of the stratum of codimension 4 and the maximal stratum in the closure of S has codimension 6.
- (5) $V = 2\mathfrak{sl}_4$. Then S is subprincipal and the maximal stratum in the closure of S has codimension 8.

For each type of Lie algebra, the subprincipal strata are among the strata of codimensions 4 and 5.

Proof. We first consider the case where $\mathrm{codim} S = 5$ and H has rank 1. We use some of the same techniques as [Kut11]. Let $v \in V$ such that $\pi(v) \in S$. Then there is a 1-parameter subgroup $\lambda: \mathbb{C}^* \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot v := v_0$ where Gv_0 is closed and Gv_0 is conjugate to H . Applying an element of G we may assume that $Gv_0 = H$. Since λ fixes v_0 , λ is a 1-parameter subgroup of H where $H^0 = \mathbb{C}^*$. Let \mathfrak{P}_λ denote the parabolic subalgebra corresponding to λ . Thus $\mathfrak{P}_\lambda = \{x \in \mathfrak{g} \mid \lim_{t \rightarrow 0} \lambda(t) \cdot x \text{ exists}\}$ so that $v \in r\mathfrak{P}_\lambda$, by definition. Hence $\pi^{-1}(S) \subset G \cdot r\mathfrak{P}_\lambda$. Now $\mathfrak{P}_\lambda \subset \mathfrak{P}_\mu$ where \mathfrak{P}_μ is a maximal parabolic corresponding to a 1-parameter subgroup μ of G . Note that any point v of $r\mathfrak{P}_\mu$ has the property that $\lim_{t \rightarrow 0} \mu(t) \cdot v \in V^\mu$.

Now let $v \in V^\lambda$ be on a closed orbit with isotropy group H . Since $V^\lambda \subset r\mathfrak{P}_\lambda \subset r\mathfrak{P}_\mu$, $\mu(t) \cdot v$ has limit a point where the isotropy group contains $\mu(\mathbb{C}^*)$. But Gv is closed, so that $\mu(\mathbb{C}^*)$ must be conjugate to $\lambda(\mathbb{C}^*)$. Hence we see that \mathfrak{P}_λ was already maximal.

Let (W, H) be the slice representation of S with nontrivial part W' . We may assume that the positive roots pair positively with λ . Let $\alpha_1, \dots, \alpha_\ell$ be the simple roots of \mathfrak{g} . Then all the root spaces $\{\mathfrak{g}_{-\alpha_i}\}_{i \in I}$ lie in \mathfrak{P}_λ where I has cardinality $\ell - 1$. Then λ is the 1-parameter subgroup which lies in the kernel of the $\{\alpha_i\}_{i \in I}$.

Now we calculate the slice representation of H , restricted to H^0 . But, up to trivial factors, this is just $(r - 1)\mathfrak{g}$, considered as a representation of H^0 . Thus this representation has at most 6 nonzero weights. But it is easy, by inspecting root systems, to show the following.

- (1) Let $\mathfrak{g} = \mathbf{A}_4$ and consider the roots restricted to a candidate for λ (i.e., three of the simple roots are trivial on λ). Then the action of λ on \mathfrak{g} has at least four pairs of nonzero weights.

- (2) Let $\mathfrak{g} = \mathbf{B}_3$ or \mathbf{C}_3 . Then the action of λ on \mathfrak{g} has at least five pairs of nonzero weights.
- (3) Let $\mathfrak{g} = \mathbf{G}_2$. Then the action of λ on \mathfrak{g} has at least five pairs of nonzero weights.

Clearly, if \mathfrak{g} has a subroot system of any type above, then any λ has too many nonzero weights. Hence we are reduced to the Lie algebras of type \mathfrak{sl}_2 , \mathfrak{sl}_3 , \mathfrak{so}_5 and \mathfrak{sl}_4 . We give the details for \mathfrak{so}_5 and leave the other cases to the reader.

The most straightforward thing to do is to find all the slice representations. We can use the techniques of [Sch78, Lemma 3.8]. Any proper slice representation of $2\mathfrak{so}_5$ is a slice representation of a slice representation at a nontrivial closed orbit of \mathfrak{so}_5 . Thus we have slice representations of the following representations.

- (1) The maximal torus acting on the root spaces where the positive root spaces are \mathfrak{g}_α , \mathfrak{g}_β , $\mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{g}_{\alpha+2\beta}$.
- (2) The group with finite cover $\mathrm{SL}_2 \times \mathbb{C}^*$ and slice representation $2\mathfrak{sl}_2 + R_1 \otimes \nu_1 + R_1 \otimes \nu_{-1}$.
- (3) The group with finite cover $\mathrm{SL}_2 \times \mathbb{C}^*$ and slice representation $2\mathfrak{sl}_2 + R_2 \otimes \nu_1 + R_2 \otimes \nu_{-1}$.

But if one takes a proper slice representation in cases (2) and (3), one ends up in the slice representation of the maximal torus T whose stratum has codimension 6. The maximal proper slice representations of T are seen to be $(3\nu_1 + 3\nu_{-1}, \mathbb{C}^*)$ and $(2\nu_1 + 2\nu_{-1} + \nu_{-2} + \nu_{-2}, \mathbb{C}^*)$. The second case is not subprincipal and has the nontrivial slice representation corresponding to the codimension four stratum. There is no stratum of codimension five with a finite isotropy group. This completes the discussion for $2\mathfrak{so}_5$ (and all other cases where $\mathrm{codim} S = 5$ and $H^0 = \mathbb{C}^*$).

If $\mathrm{codim} S = 5$ and the isotropy group is finite, then by Corollary 7.4 we have a stratum S' of codimension 4 which is subprincipal. Thus we need only consider the case where $\mathrm{codim} S = 4$ and H is a finite subgroup of the maximal torus T of G . Applying the roots to H we see that we must have at most two roots γ such that γ is nontrivial on H . If \mathfrak{g} contains a copy of \mathfrak{sl}_4 , then we have at most two simple roots α and β which are nontrivial on H . If two simple roots α and β are nontrivial on H and the third simple root γ is trivial, then $\alpha + \gamma$ or $\beta + \gamma$ is a root and we get at least six roots nontrivial on H . If only one simple root is nontrivial on H , one easily sees that there are at least 8 roots which are not trivial on H . Similar calculations rule out root systems of rank at least three and of type \mathbf{G}_2 . By calculating all the slice representations in the remaining cases one sees that only for $2\mathfrak{so}_5$ do we get an S of codimension 4. This completes our proof. \square

Let $V = \bigoplus_{i=1}^d r_i \mathfrak{g}_i$, G and $Z = Z_1 \times \cdots \times Z_d$ be as in the beginning of this section. Let $\varphi \in \mathrm{Aut}(Z)$. Then by [Sch13] we know that φ permutes the strata of the same codimension. The strata of Z are products $S_1 \times \cdots \times S_d$ where S_i is a stratum of Z_i . If $S := S_{i_1} \times \cdots \times S_{i_r}$ is a product of strata where $S_{i_j} \subset Z_{i_j}$, we let S' denote the product of S with the principal strata of the remaining Z_k . We say that a stratum S of Z is *simple* if it is S'_i for some $S_i \subset Z_i$, otherwise we say that S is *composite*. By Corollary 7.5 the strata of codimension at most 5 are simple, hence they are permuted by φ .

Lemma 7.11. *Let S be a stratum of codimension c at most 9. Then S is simple if and only if $\varphi(S)$ is simple.*

Proof. We proceed by induction on c where the lemma holds if $c \leq 5$. If S is composite, then without loss of generality we have that $S = (S_1 \times T_2)'$ or $S = (R_1 \times S_2 \times T_3)'$ where the codimension of the strata adds up to c . We leave the case of the product of three codimension 3 strata to the reader. Let us do the case where $\mathrm{codim} S_1 = 3$ and $\mathrm{codim} T_2 = c - 3 \leq 6$. Then for some i , $\varphi(S'_1) = S'_i$ where $\mathrm{codim} S_i = 3$ in Z_i . Similarly, $\varphi(T'_2) = T'_j$ where $\mathrm{codim} T_j = c - 3$. If $i = j$, then by Remark 7.7 we would have that T'_j is in the closure of S'_i . Applying φ^{-1} we would get that T'_2 is in the closure of S'_1 which is absurd. Hence $i \neq j$. Now $(S_1 \times T_2)'$ is the maximal dimensional stratum in $\overline{S'_1} \cap \overline{T'_2}$ and similarly for $(S_i \times T_j)'$. It follows that φ sends

$(S_1 \times T_2)'$ onto $(S_i \times T_j)'$. Hence S composite implies that $\varphi(S)$ is composite. Using the same argument for φ^{-1} we get the converse. \square

Corollary 7.12. *Let $\varphi \in \text{Aut}(Z)$. Then there is a σ -equivariant $\Psi \in \text{GL}(V)$ inducing $\psi \in \text{Aut}(Z)$ such that $\psi^{-1} \circ \varphi$ preserves all the strata of codimension at most 5.*

Proof. First suppose that $S = S'_1$ where S_1 has codimension 3 and $r_1 \mathfrak{g}_1 = 2\mathfrak{sl}_3$. Then there is a stratum T_1 of codimension 4 in $\overline{S_1}$. Now $\varphi(S) = S'_i$ where S_i has codimension 3. We may suppose that $i = 2$. Then $\varphi(T'_1)$ is a simple stratum of codimension 4 which is in the closure of S'_2 . The only possibility is that $\varphi(T'_1)$ is T'_2 where T_2 is a codimension 4 stratum of Z_2 . It follows from Remark 7.7 that $r_2 \mathfrak{g}_2 = 2\mathfrak{sl}_3$. Continuing inductively and perhaps renumbering the factors of \mathfrak{g} , we find that $r_i \mathfrak{g}_i \simeq 2\mathfrak{sl}_3$ for $1 \leq i \leq k$ where $\varphi(S'_i) = S'_{i+1}$ and $\varphi(T'_i) = T'_{i+1}$ for $1 \leq i < k$ and $\varphi(S'_k) = S'_1$ and $\varphi(T'_k) = T'_1$. Here the S_i have codimension 3 and the T_i have codimension 4. There is an obvious σ -equivariant element $\Psi \in \text{GL}(V)$ such that the induced mapping ψ has the same effect on the S'_i and T'_i as φ , $1 \leq i \leq k$. Moreover, ψ is the identity on all other strata of codimension at most 5. Thus $\psi^{-1} \circ \varphi$ preserves the S_i and T_i for $1 \leq i \leq k$. We could have that there is no codimension four stratum in Z_1 , in which case we have $r_1 \mathfrak{g}_1 \simeq 3\mathfrak{sl}_2$ and we continue as before. The other possibilities for S_1 follow the same pattern using the information about the strata of codimension at most 9 in Remark 7.7 and Proposition 7.10. Thus we may reduce to the case that φ preserves the strata S'_i of codimension at most 5 for $1 \leq i \leq k$. One then proceeds by the obvious induction. \square

Let $E, F, M, \mathcal{E}, \mathcal{F}, \mathcal{G}$, etc. be as in §6. Then we have condition (*) and by [Sch80, Proposition 9.3, Theorem 10.7] we also have (**). Recall that V^\dagger is the inverse image of the strata of codimension at most 5 and that $V \setminus V^\dagger$ has codimension 4. Our aim now is to establish that

$$(***) \quad 0 \rightarrow \pi^\# \varphi^* \mathcal{E} \rightarrow \pi^\# \varphi^* \mathcal{F} \rightarrow \pi^\# \mathcal{G} \rightarrow 0$$

is exact on V^\dagger . Suppose this is true. Then (6.2.4) and (6.2.5) hold by the same argument as before: one takes cohomology over V^\dagger and uses the fact that M has depth 3 for the ideal of $V \setminus V^\dagger$. One concludes that $\Gamma(V, \pi^\# \varphi^* \mathcal{F}')$ is a free $\mathcal{O}(V)$ -module which implies that there is a lift Φ of φ over V_{pr} , hence over V .

Lemma 7.13. *The module M is reflexive.*

Proof. It is enough to show that $M = M^{**}$ over an open subset $V^b \subset V$ whose complement has codimension 3. Set $V^b := \{v \in V \mid \mathfrak{g}_v = 0\}$. Over the open set V^b , M is locally free, hence isomorphic to M^{**} , so we need only prove that $V \setminus V^b$ has codimension at least 3. Now the inverse image of the strata of Z of codimension at least four has codimension three in V , so we only need worry about the inverse image of the strata of codimension three. Using the slice theorem, it is enough to show that W^b has codimension three in W where W is an $H = \mathbb{C}^*$ module such that $W' = 2\nu_1 + 2\nu_{-1}$. But \mathfrak{h} only vanishes at the origin of W' . Hence we have codimension four. \square

Proposition 7.14. *Let V be a 2-principal G -module and S a stratum of Z with isotropy group H where H^0 is a torus. Let $v \in \pi^{-1}(S)$ and let $\varphi \in \text{Aut}(Z)$ such that φ preserves S .*

- (1) *There is a neighborhood N_z of $z := \pi(v) \in Z$ (classical topology) and a biholomorphic lift Φ of φ sending $\pi^{-1}(N_z)$ onto $\pi^{-1}(\varphi(N_z))$.*
- (2) *Let $v' = \Phi(v)$. Let U (resp. U') be a Zariski open neighborhood of v (resp. v'). Then $\pi^{-1}(\varphi(\pi(U)))$ (resp. $\pi^{-1}(\varphi^{-1}(\pi(U'))))$ contains a neighborhood of v' (resp. v) in the Zariski topology.*

Proof. Suppose that we have (1). Let U be a neighborhood of v in the Zariski topology. Then $\Phi(\pi^{-1}(N_z) \cap U) \subset \pi^{-1}(\varphi(N_z \cap \pi(U)))$ so that $v' = \Phi(v)$ is an interior point of the constructible

set $C := \pi^{-1}(\varphi(\pi(U)))$. Removing the boundary we obtain a Zariski open neighborhood of v' contained in C . Using Φ^{-1} we get the analogous property for neighborhoods of v' . Hence we only have to produce the local lift Φ of φ .

Since φ preserves S , the holomorphic slice theorem gives us an equivariant biholomorphism Ψ of $\pi^{-1}(N_{\varphi(z)})$ with $\pi^{-1}(N_z)$ where N_z is a neighborhood of z and $N_{\varphi(z)}$ is a neighborhood of $\varphi(z)$. Let $\psi: N_{\varphi(z)} \rightarrow N_z$ be the induced map. We can arrange that $\psi(\varphi(z)) = z$. By Proposition 5.4 we can lift $\psi \circ \varphi$ over a neighborhood N'_z of z . Composing with Ψ^{-1} we find a local lift of φ as required, and we have (1). \square

Proof of Theorem 1.14. We can reduce to the case that φ fixes each stratum of codimension at most 5. It is enough to show that (***) is exact on V^\dagger . Let \mathcal{L} be one of \mathcal{E} , \mathcal{F} or \mathcal{G} . Let $v \in V$ and define

$$\tilde{\mathcal{L}}_v = \lim_{U \ni v} \mathcal{L}(\pi(V_{\text{pr}} \cap U))$$

where U ranges over Zariski open subsets of V containing v . Then $\tilde{\mathcal{L}}_v$ is an $\mathcal{O}_{Z,z}$ -module where $z = \pi(v)$ and we have that $(\pi^\# \mathcal{L})_v = \mathcal{O}_{V,v} \otimes_{\mathcal{O}_{Z,z}} \tilde{\mathcal{L}}_v$. Suppose that $v \in \pi^{-1}(S)$ where $\text{codim } S \leq 5$. By Proposition 7.14 there is a local lift Φ of φ in a saturated neighborhood of v and we have that $(\varphi^* \mathcal{L})_v \simeq \mathcal{O}_{Z,z} \otimes_{\mathcal{O}_{Z,z'}} \tilde{\mathcal{L}}_{v'}$ where $v' = \Phi(v)$ and $z' = \varphi(z)$. Thus $(\pi^\# \varphi^* \mathcal{L})_v \simeq \mathcal{O}_{V,v} \otimes_{\mathcal{O}_{Z,z'}} \tilde{\mathcal{L}}_{v'}$ where the map sending v to z' is $\varphi \circ \pi$. Now from Φ we get an isomorphism of the completions $\hat{\mathcal{O}}_{V,v'} \simeq \hat{\mathcal{O}}_{V,v}$ which lies over the isomorphism $\hat{\varphi}: \hat{\mathcal{O}}_{Z,z'} \simeq \hat{\mathcal{O}}_{Z,z}$. If the sequence (***) failed to be exact at v , then the sequence

$$(7.14.1) \quad 0 \rightarrow (\pi^\# \mathcal{E})_{v'} \rightarrow (\pi^\# \mathcal{F})_{v'} \rightarrow (\pi^\# \mathcal{G})_{v'} \rightarrow 0$$

tensoring with $\hat{\mathcal{O}}_{V,v'} \simeq \hat{\mathcal{O}}_{V,v}$, would fail to be exact at v' . Since tensoring with the completion of a local ring is faithfully flat, we would find that (7.14.1) is not exact at v' . So it is enough to show that (7.14.1) is exact. But as we saw in §6, this is equivalent to M being reflexive, which is Lemma 7.13. \square

8. ACTIONS OF COMPACT GROUPS

Some things become simpler when we are dealing with compact group actions. Let K be a compact Lie group and W a real K -module. Then we can put a smooth structure on the quotient W/K (see [Sch75]). Simply put, a function on W/K is smooth if it pulls back to a smooth (necessarily K -invariant) smooth function on W . We can make this more concrete, as follows. Let p_1, \dots, p_d be homogeneous generators of $\mathbb{R}[W]^K$ and let $p = (p_1, \dots, p_d): W \rightarrow \mathbb{R}^d$. Now p is proper and separates the K -orbits in W . Let X denote $p(W) \subset \mathbb{R}^d$. Then X is a closed semialgebraic set and p induces a homeomorphism $\bar{p}: W/K \rightarrow X$. The main theorem of [Sch75] says that $p^* C^\infty(X) = C^\infty(W)^K$ where a function f on X is smooth if it extends to a smooth function in a neighborhood of X . Now we can define the notion of a smooth mapping $X \rightarrow X$ (or $W/K \rightarrow W/K$) in the obvious way. We also see an \mathbb{R}^* -action on X where $t \cdot (y_1, \dots, y_d) = (t^{e_1} y_1, \dots, t^{e_d} y_d)$ where $t \in \mathbb{R}^*$, $(y_1, \dots, y_d) \in X$ and e_j is the degree of p_j , $j = 1, \dots, d$. Or one can just say that \mathbb{R}^* acts on $W/K \simeq X$ where $t \cdot p(w) = p(tw)$, $t \in \mathbb{R}^*$, $w \in W$.

We have the complexification $G = K_{\mathbb{C}}$ acting on $V = W \otimes_{\mathbb{R}} \mathbb{C}$. Then our generators p_i of $\mathbb{R}[W]^K$, considered as polynomials on V , generate $\mathcal{O}(V)^G$. We say that W is *admissible* if V is 2-principal. We don't need the second condition of Definition 1.1 because of the following result which can be found in [Sch09, Lemma 2.4] (stated for finite groups, but the proof works for compact groups as well).

Lemma 8.1. *Let $\varphi: W/K \rightarrow W/K$ be a smooth isomorphism. Then $\lim_{t \rightarrow 0} (t^{-1} \circ \varphi \circ t)(w)$ exists for every $w \in W$ and the resulting map $\varphi_0: W/K \rightarrow W/K$ is a quasilinear isomorphism.*

Let us say that an admissible W has the *lifting property* if every quasilinear isomorphism of W/K has a linear lift to W , which has to be invertible by Proposition 2.6. Equivalently, $N_{\mathrm{GL}(W)}(K)$ maps onto $\mathrm{Aut}_{\mathrm{ql}}(W/K)$.

Theorem 8.2. *Let $\varphi: W/K \rightarrow W/K$ be a diffeomorphism where W has the lifting property. Then there is a lift $\Phi: W \rightarrow W$ which is σ -equivariant for some $\sigma \in \mathrm{Aut}(K)$.*

Proof. We have an isotopy φ_t where $\varphi_1 = \varphi$ and φ_0 is quasilinear. But hypothesis there is a σ -equivariant lift $\Phi_0: W \rightarrow W$ of φ_0 . Thus it suffices to equivariantly lift $\varphi_0^{-1} \circ \varphi$. But this is a consequence of the isotopy lifting theorem of [Sch80]. \square

Proposition 8.3. *If V is 2-principal and has the lifting property, then so does W .*

Proof. Let $\varphi \in \mathrm{Aut}_{\mathrm{ql}}(W/K)$. Then clearly φ extends to an element (which we also call φ) of $\mathrm{Aut}_{\mathrm{ql}}(Z)$. By hypothesis there is a lift $\Phi \in \mathrm{GL}(V)$ of φ and by Proposition 2.6 Φ normalizes G . As in Remark 2.7, we may change Φ by an arbitrary inner automorphism of G . Since $\Phi K \Phi^{-1}$ is conjugate to K we can reduce to the case that $\Phi \in N_G(K)$. Since φ is real, $\bar{\Phi}$ covers $\bar{\varphi} = \varphi$ as well. Since K is fixed by complex conjugation, $\bar{\Phi}$ also normalizes K . Then $\Phi^{-1}\bar{\Phi}$ normalizes K and induces the identity on Z . By Proposition 2.6, $\Phi^{-1}\bar{\Phi}$ is multiplication by an element $g \in G$ and $g \in N_G(K)$. We have that

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus i\mathfrak{z}(\mathfrak{k}) \oplus [\mathfrak{k}, \mathfrak{k}] \oplus i[\mathfrak{k}, \mathfrak{k}]$$

where $\mathfrak{z}(\mathfrak{k})$ denotes the center of \mathfrak{k} . Since no nonzero element of $i[\mathfrak{k}, \mathfrak{k}]$ can normalize $[\mathfrak{k}, \mathfrak{k}]$, the Lie algebra of $N_G(K)$ is $\mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}$ and K is a maximal compact subgroup of $N_G(K)$ since it is already a maximal compact subgroup of G . Thus our element g can be written as kp where $k \in K$, $p = \exp(iA)$ and $A \in \mathfrak{z}(\mathfrak{k})$. Let $q = \exp(iA/2)$ and set $\Phi' := \Phi q$. Then Φ' is still a lift of φ and $(\Phi')^{-1}\bar{\Phi}' = q^{-1}kpq^{-1} = k$. Thus $\bar{\Phi}' = \Phi'k$ so that Φ' has real entries. Hence Φ' is a lift of φ to W and W has the lifting property. \square

Theorem 1.15 of the Introduction is a consequence of Theorem 8.2 and Proposition 8.3.

Many of the examples in section §4 (Theorem 1.8 and Remark 1.9) apply to the case of representations of compact groups. Let $G_2(\mathbb{R})$ and $B_3(\mathbb{R})$ denote the compact forms of G_2 and B_3 which act on \mathbb{R}^7 and \mathbb{R}^8 , respectively. Then the representations $(4\mathbb{R}^7, G_2(\mathbb{R}))$, $(5\mathbb{R}^8, B_3(\mathbb{R}))$ and $(2\mathbb{C}^2, \mathrm{SU}_2(\mathbb{C}))$ do not have the lifting property. On the other hand, by Proposition 8.3, we have many examples (W, K) which have the lifting property.

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